Revisit Coin-Change Problem

- So far we’ve tried: A greedy algorithm that does not work for all inputs (it is incorrect)
- New tricks we’ve learned...
  - Is there an exhaustive search algorithm?

```python
def exhaustiveChange(amount, denominations):
    bestN = 100
    count = [0 for i in xrange(len(denominations))]
    while True:
        for i, coinValue in enumerate(denominations):
            count[i] += 1
            if (count[i] * coinValue < 100):
                break
            count[i] = 0
        n = sum(count)
        if n == 0:
            break
        value = sum([count[i] * denominations[i] for i in xrange(len(denominations))])
        if (value == amount):
            if (n < bestN):
                bestN = n
    return bestN
```

How many coin combinations does this routine test?
Revisit Coin-Change Problem

- Other Tricks? A branch-and-bound algorithm

```python
def branchAndBoundChange(amount, denominations):
    bestN = amount
    count = [0 for i in range(len(denominations))]
    while True:
        for i, coinValue in enumerate(denominations):
            count[i] += 1
            if (count[i]*coinValue < amount):
                break
            count[i] = 0
        n = sum(count)
        if n == 0:
            break
        if n > bestN:
            continue
        value = sum([count[i]*denominations[i] for i in range(len(denominations))])
        if value == amount:
            if n < bestN:
                bestN = n
    return bestN
```

- Correct, and works well for most cases, but might be as slow as an exhaustive search for some inputs.

- Is there anything else we can try?

Tabulating the Answers

- If it is costly to compute the answer for a given input, then there may be advantages to caching the result of previous calculations in a table
- This trades-off time-complexity for space
- How could we fill in the table in the first place?
  - Run our best correct algorithm
  - Can the table itself be used to speed up the process?
Solutions using a Table

• Suppose you are asked to fill-in the unknown table entry for 67¢
• It must differ from previous known optimal result by at most one coin…

• So what are the possibilities?
  – BestChange(67¢) = 25¢ + BestChange(42¢), or
  – BestChange(67¢) = 20¢ + BestChange(47¢), or
  – BestChange(67¢) = 10¢ + BestChange(57¢), or
  – BestChange(67¢) = 5¢ + BestChange(62¢), or
  – BestChange(67¢) = 1¢ + BestChange(66¢)

A Recursive Coin-Change Algorithm

```python
def RecursiveChange(M, c):
    if (M == 0):
        return [0 for i in xrange(len(c))]
    smallestNumberOfCoins = M+1
    for i in xrange(len(c)):
        if (M >= c[i]):
            thisChange = RecursiveChange(M - c[i], c)
            thisChange[i] += 1
            if (sum(thisChange) < smallestNumberOfCoins):
                bestChange = thisChange
                smallestNumberOfCoins = sum(thisChange)
    return bestChange
```

• The only problem is… this is still too slow
• Let’s see why…
Recursion Recalculations

- We saw this before with RecursiveFibonacci()
- Recursion often results in many redundant calls
- Even after only two levels of recursion 6 different change values are repeated multiple times
- How can we avoid this repetition?
- Cache precomputed results in a table!

\[
\begin{align*}
\text{Change}(40) & = 25 + \text{Change}(15) \\
& = 25 + 10 + \text{Change}(5) \\
& = 25 + 5 + \text{Change}(10) \\
\end{align*}
\]

Change(20) =
\[
\begin{align*}
20 + \text{Change}(10) \\
& = 20 + 10 + \text{Change}(0) \\
& = 20 + 5 + \text{Change}(15) \\
\end{align*}
\]

Change(10) =
\[
\begin{align*}
10 + \text{Change}(5) \\
& = 10 + 5 + \text{Change}(15) \\
& = 10 + 10 + \text{Change}(20) \\
& = 10 + 5 + \text{Change}(25) \\
\end{align*}
\]

Change(5) =
\[
\begin{align*}
5 + \text{Change}(3) \\
& = 5 + 25 + \text{Change}(15) \\
& = 5 + 20 + \text{Change}(10) \\
& = 5 + 5 + \text{Change}(30) \\
\end{align*}
\]

Back to Table Evaluation

- When do we fill in the values of the table?
- We could do it lazily as needed... as each call to BestChange() progresses from M down to 1
- Or we could do it from the bottom-up – tabulating all values from 1 up to M
- Thus, instead of just trying to find the minimal number of coins to change M cents, we attempt the solve the superficially harder problem of solving for the optimal change for all values from 1 to M

\[
\begin{align*}
1\text{c} & = [0,0,0,0,1] \\
2\text{c} & = [0,0,0,0,2] \\
3\text{c} & = [0,0,0,0,3] \\
\cdots \\
M\text{c} & = [?,?,?,?,?] \\
\end{align*}
\]
Change via Dynamic Programming

```python
def DPChange(M, c):
    change = [[0 for i in xrange(len(c))]]
    for m in xrange(1,M+1):
        bestNumCoins = m+1
        for i in xrange(len(c)):
            if m >= c[i]:
                thisChange = [x for x in change[m - c[i]]]
                thisChange[i] += 1
                if sum(thisChange) < bestNumCoins:
                    change[m:m] = [thisChange]
                    bestNumCoins = sum(thisChange)
    return change[M]
```

- Recall, BruteForceChange() was $O(M^d)$
- DPChange() is $O(Md)$

Dynamic Programming

- *Dynamic Programming* is a technique for computing recurrence relations efficiently by storing partial or intermediate results

- Three keys to constructing a dynamic programming solution:
  1. Formulate the answer as a recurrence relation
  2. Consider all instances of the recurrence at each step
  3. Order evaluations so you will always have precomputed the needed partial results
Yet Another DP Example
Manhattan Tourist Problem (MTP)

Imagine seeking a path from source to destination in a Manhattan-like city grid that maximizes the number of attractions (*) passed. With the following caveat— at every step you must make progress towards the goal.

We treat the city map as a graph, with a “vertices” at each corner, and weighted edges along each block. The weights are the number of attractions along each block.

Manhattan Tourist Problem: Formulation

Goal: Find the maximum weighted path in a grid.

Input: A weighted grid \( G \) with two distinct vertices, one labeled “source” and the other labeled “destination”

Output: A longest path in \( G \) from “source” to “destination”
MTP: Greedy Algorithm Is Not Optimal

Greedy Algorithm:
At each step select the maximum weight block.

Greed has a short horizon

promising start, but leads to bad choices!

MTP as a Dynamic Program
MTP Strategy

• Instead of solving the Manhattan Tourist problem directly, (i.e. the path from (0,0) to (n,m)) we will solve a more general problem: find the longest path from (0,0) to any arbitrary vertex (i,j).

• If the longest path from (0,0) to (n,m) passes through some vertex (i,j), then the path from (0,0) to (i,j) must be the longest. Otherwise, you could increase your path by changing it.

MTP: Simple Recursive Program

What's wrong with this approach?

\[
\text{MT}(n,m) \\
\begin{align*}
\text{if } n = 0 \text{ and } m = 0 & \quad \text{return } 0 \\
\text{if } n = 0 & \quad \text{return } \text{MT}(0,m-1) + \text{len(edge)} \text{ from } (0,m-1) \text{ to } (0,m) \\
\text{if } m = 0 & \quad \text{return } \text{MT}(n-1, 0) + \text{len(edge)} \text{ from } (n-1,0) \text{ to } (n,0) \\
\text{x } \leftarrow & \quad \text{MT}(n-1,m) + \text{len(edge)} \text{ from } (n-1,m) \text{ to } (n,m) \\
\text{y } \leftarrow & \quad \text{MT}(n,m-1) + \text{len(edge)} \text{ from } (n,m-1) \text{ to } (n,m) \\
\text{return } & \quad \text{max}\{x,y\}
\end{align*}
\]
MTP: Ordering Evaluations

- Calculate optimal path score for each vertex in the graph
- Each vertex’s score is the maximum of the prior vertices score plus the weight of the connecting edge in between

MTP: Dynamic Programming (cont’d)

First, fill in the easy ones!

Then grow the solution a block at a time while tabulating the results for each intersection

Note: We’ll use our table to keep track of two things. The value of the best path to the given intersection, and the direction from where it came
MTP: Dynamic Programming (cont'd)

Keep growing…
(3 blocks)

MTP: Dynamic Programming (cont’d)

And growing…
(4 blocks)
MTP: Dynamic Programming (cont’d)

Once the "destination" node (intersection) is reached, we're done.

Our table will have the answer of the maximum number of attractions stored in the entry associated with the destination.

We use the "links" back in the table to recover the path.
(Backtracking)
MTP: Recurrence

Computing the score for a point \((i,j)\) by the recurrence relation:

\[
s_{i,j} = \max \begin{align*}
& s_{i-1,j} + \text{weight of the edge between } (i-1, j) \text{ and } (i, j) \\
& s_{i,j-1} + \text{weight of the edge between } (i, j-1) \text{ and } (i, j)
\end{align*}
\]

The running time is \(n \times m\) for a \(n\) by \(m\) grid

(You visit all intersections once, and performed 2 tests)

\((n = \# \text{ of rows}, m = \# \text{ of columns})\)

MTP Pseudocode

\begin{verbatim}
MANHATTANTOURIST(\(\overrightarrow{w}, \overrightarrow{w}, n, m\))
1  \(s_{0,0} \leftarrow 0\)
2  \text{for } i \leftarrow 1 \text{ to } n
3     \(s_{i,0} \leftarrow s_{i-1,0} + \overrightarrow{w}_{i,0}\)
4  \text{for } j \leftarrow 1 \text{ to } m
5     \(s_{0,j} \leftarrow s_{0,j-1} + \overrightarrow{w}_{0,j}\)
6  \text{for } i \leftarrow 1 \text{ to } n
7      \text{for } j \leftarrow 1 \text{ to } m
8          \(s_{i,j} \leftarrow \max \begin{align*}
& s_{i-1,j} + \overrightarrow{w}_{i,j} \\
& s_{i,j-1} + \overrightarrow{w}_{i,j}
\end{align*}\)
9  \text{return } s_{n,m}
\end{verbatim}
MTP Python Code

# Dynamic programming solution to the Manhattan Tourist Problem
# h = horizontal weights; h[i][j] = attractions between h[i][j-1] and h[i][j]
# v = vertical weights; v[i][j] = attractions between v[i-1][j] and v[i][j]
# rows and columns are the number of rows and columns in the grid

def ManhattanTourist(v, h, rows, columns):
    # Set up an empty rows x columns matrix.
    MT = []
    for r in range(0, rows):
        MT.append([0] * columns)

    # Fill in the first row and first column.
    MT[0][0] = 0
    for r in range(1, rows):
        MT[r][0] = MT[r-1][0] + v[r][0]
    for c in range(1, columns):
        MT[0][c] = MT[0][c-1] + h[0][c]

    # Then fill in the rest of the table.
    for r in range(1, rows):
        for c in range(1, columns):
            MT[r][c] = max(MT[r-1][c] + v[r][c], MT[r][c-1] + h[r][c])
    return MT[rows-1][columns-1]

Manhattan Is Not A Perfect Grid

What about diagonals? Broadway, Greenwich, etc.

- Easy to fix. Just adds more recursion cases.
- The score at point B is given by:

\[
s_B = \max \begin{cases}
    s_{A_1} + \text{weight of the edge } (A_1, B) \\
    s_{A_2} + \text{weight of the edge } (A_2, B) \\
    s_{A_3} + \text{weight of the edge } (A_3, B)
\end{cases}
\]