An **Eulerian path** (cycle) in a graph $G$ is a path (cycle) that visits every edge in $G$ exactly once.

A vertex is **balanced** if it has the same number of incoming and outgoing edges.

A vertex is **semi-balanced** if indegree differs by outdegree by 1.

**Theorem**: A connected directed graph $G$ contains an Eulerian cycle if and only if all the vertices are balanced.

**Proof.** This is a just a proof-sketch!!!

$(\Rightarrow)$ Suppose $G$ contains an Eulerian cycle. We need to show that all vertices in $G$ are balanced.

For any vertex $v$ and any edge entering $v$, there is a corresponding edge leaving $v$. In particular, these are the edges used in the Eulerian cycle. Thus $\text{in}(v) = \text{out}(v)$.

$(\Leftarrow)$ Suppose all vertices in $G$ and balanced. We need to show that $G$ contains a Eulerian cycle.

We will do this by showing how to construct such a cycle.

- **Step 1**: Start at some vertex $v$. Keep traveling on unused edges from $v$ until you hit a dead end. We claim that the dead end must be at $v$ since the graph is balanced. This produces a cycle.
- **Step 2**: If the cycle we just found is not Eulerian, then it must contain some vertex $w$ with untraversed edges (because the graph is connected). Repeat step 1 from $w$.
- **Step 3**: Combine all cycles found into one Eulerian cycle.
Theorem: A connected, directed graph $G$ has an Eulerian path if and only if it contains at most two semi-balanced vertices and all other vertices are balanced.

- The path must start and end at the semi-balanced vertices.
- One vertex $v$ has $in(v) - out(v) = 1$ and one vertex $w$ must have $out(w) - in(w) = 1$
- Add edge from $v$ to $w$ and search for Eulerian cycle. Remove the edge $(v, w)$ to get Eulerian path.