COMP 355
Advanced Algorithms

Clique, Vertex Cover, and Dominating Set

Chapter 8 (KT)

Section 34.5(CLRS)

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Recap

Last time we gave a reduction from 3SAT (satisfiability of boolean formulas in 3-CNF form) to IS (independent set in graphs).

Recall that to show that a decision problem (language) L is NP-complete we need to show:

1. $L \in NP$.
2. L is NP-hard

Some NP-Complete Problems

Clique (CLIQUE): The clique problem is: given an undirected graph G = (V,E) and an integer *k*, does G have a subset V′ of k vertices such that for each distinct u, v ∈ V′, (u, v) ∈ E. In other words, does G have a *k* vertex subset whose induced subgraph is complete?

Vertex Cover (VC): A vertex cover in an undirected graph G = (V,E) is a subset of vertices V′ ⊆ V such that every edge in G has at least one endpoint in V′. The vertex cover problem (VC) is: given an undirected graph G and an integer *k*, does G have a vertex cover of size *k*?

Dominating Set (DS): A dominating set in a graph G = (V,E) is a subset of vertices V′ such that every vertex in the graph is either in V′ or is adjacent to some vertex in V′. The dominating set problem (DS) is: given a graph G = (V,E) and an integer *k*, does G have a dominating set of size *k*?

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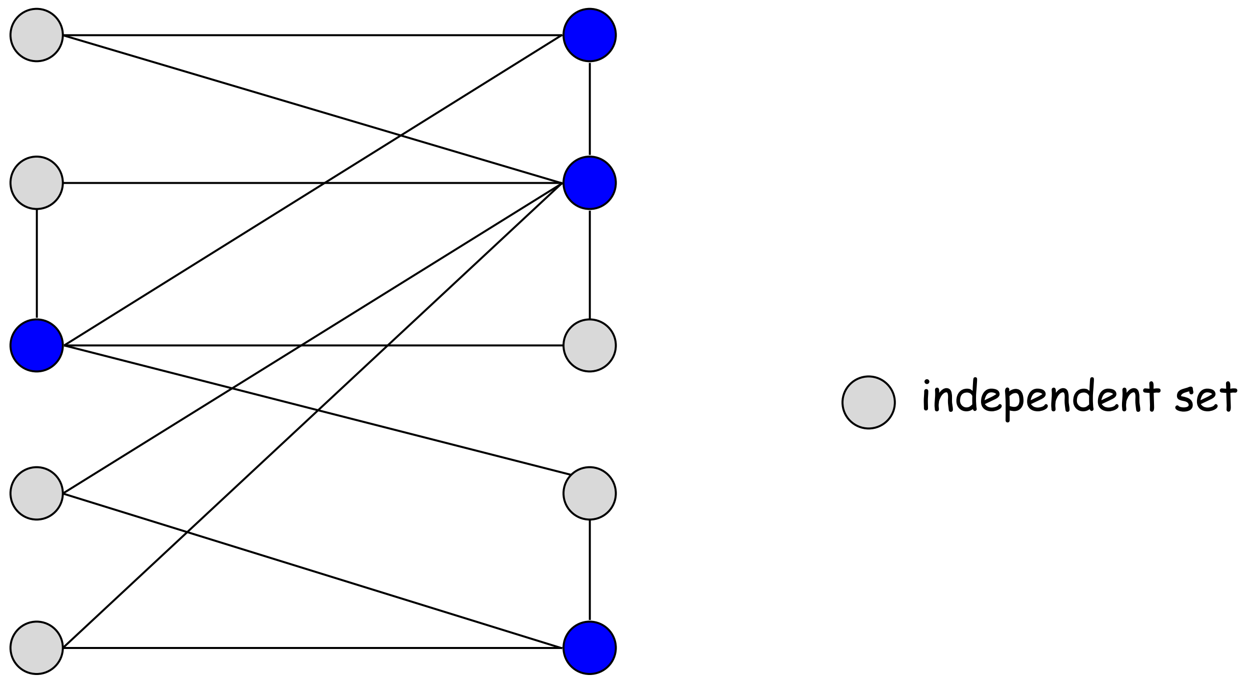
Independent Set

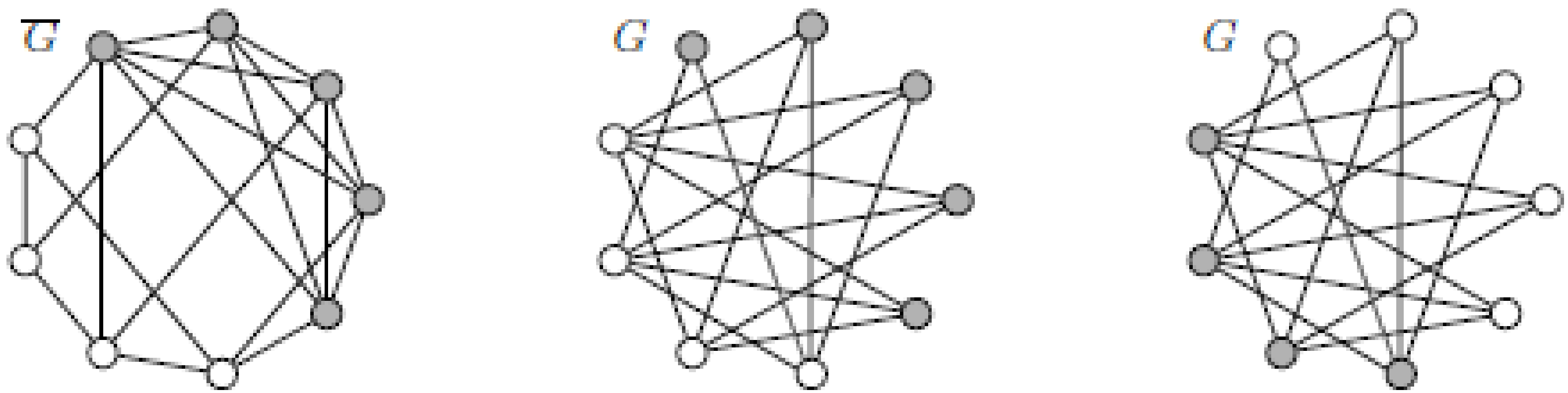
INDEPENDENT SET: Given a graph G = (V, E) and an integer k, is there a subset of vertices $S \subseteq V$ such that $|S| \geq k$, and for each edge at most one of its endpoints is in S?

Ex. Is there an independent set of size ≥ 6? Yes.

Ex. Is there an independent set of size ≥ 7? No.



Clique, Independent set, and Vertex Cover



V' is a clique of size k in $\overline{G}$ $\Leftrightarrow$ V' is a independent set of size k in G $\Leftrightarrow$ $V \setminus V'$ is a vertex cover of size $n - k$ in G

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Clique, Independent set, and Vertex Cover

Lemma: Given an undirected graph $G = (V,E)$ with n vertices and a subset $V' \subseteq V$ of size k. The following are equivalent:

i. V' is a clique of size k for the complement, G

ii. V' is an independent set of size k for G

iii. $V \setminus V'$ is a vertex cover of size $n - k$ for G, (where $n = |V|$)

Proof:

(i) ⇒ (ii): If V′ is a clique for G, then for each $u, v \in V'$, $\{u, v\}$ is an edge of G implying that $\{u, v\}$ is not an edge of G, implying that V′ is an independent set for G.

(ii) ⇒ (iii): If V′ is an independent set for G, then for each $u, v \in V'$, $\{u, v\}$ is not an edge of G, implying that every edge in G is incident to a vertex in $V \setminus V'$, implying that $V \setminus V'$ is a vertex cover for G.

(iii) ⇒ (i): If $V \setminus V'$ is a vertex cover for G, then for any $u, v \in V'$ there is no edge $\{u, v\}$ in G, implying that there is an edge $\{u, v\}$ in G, implying that V′ is a clique in G.

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CLIQUE is NP-Complete

Theorem: CLIQUE is NP-complete.

CLIQUE ∈ NP: We guess the k vertices that will form the clique. We can easily verify in polynomial time that all pairs of vertices in the set are adjacent (e.g., by inspection of $O(k^2)$ entries of the adjacency matrix).

IS $\leq_P$ CLIQUE: We want to show that given an instance of the IS problem (G, k), we can produce an equivalent instance of the CLIQUE problem in polynomial time. The reduction function f inputs G and k, and outputs the pair (G, k). Clearly this can be done in polynomial time. By the above lemma, this instance is equivalent.

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VC is NP-complete

Theorem: VC is NP-complete.

VC ∈ NP: The certificate consists of the k vertices in the vertex cover. Given such a certificate we can easily verify in polynomial time that every edge is incident to one of these vertices.

IS $\leq_P$ VC: We want to show that given an instance of the IS problem (G, k), we can produce an equivalent instance of the VC problem in polynomial time. The reduction function f inputs G and k, computes the number of vertices, n, and then outputs (G, n − k). Clearly this can be done in polynomial time.

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Dominating Set (Definition)

Problem:

- Dominating-set = Given <*G*, *k*>, does a dominating set of size (at most) *k* for *G* exists?

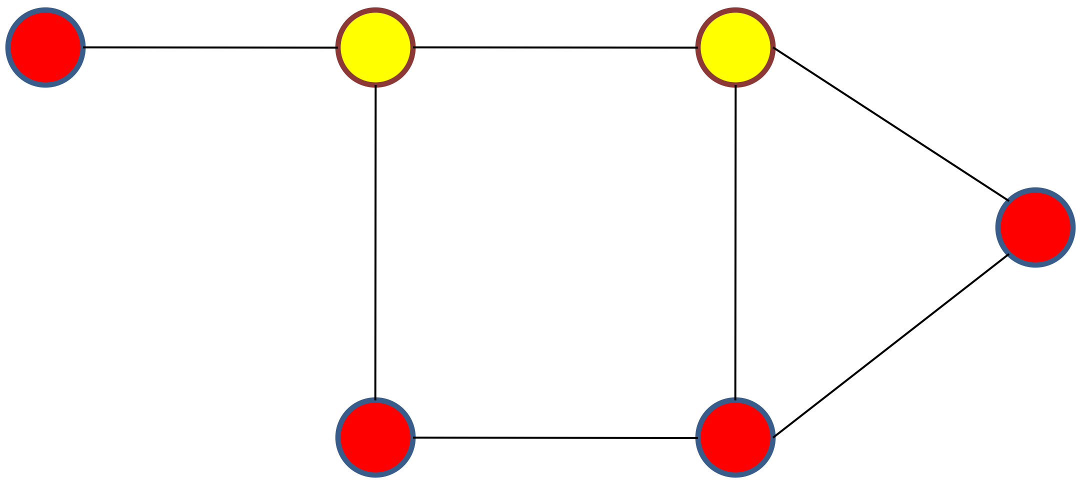
Let *G*=(*V*,*E*) be an undirected graph

A dominating set *D* is a set of vertices that covers all vertices

- i.e., every vertex of *G* is either in *D* or is adjacent to at least one vertex from *D*

Dominating Set (Example)

Size-2 example : {Yellow vertices}



Dominating Set (Proof Sketch)

Steps:

1) Show that Dominating-set $\in$ NP.
2) Show that Dominating-set is not easier than a NPC problem
   - We choose this NPC problem to be Vertex cover
   - Reduction from Vertex-cover to Dominating-set
3) Show the correspondence of “yes” instances between the reduction

Dominating Set - (1) NP

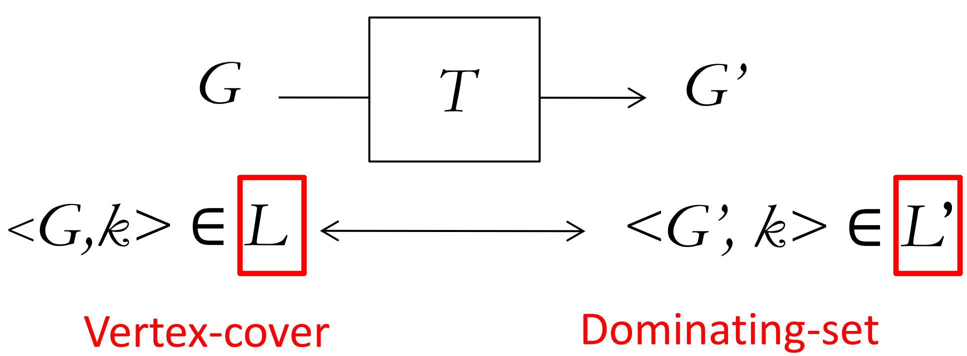
It is trivial to see that Dominating-set ∈ NP

- Given a vertex set D of size *k*, we check whether (V-D) are adjacent to D
- i.e., for each vertex, v, in D, whether v is adjacent to some vertex u in D

Dominating Set - (2) Reduction

Reduction - Graph transformation

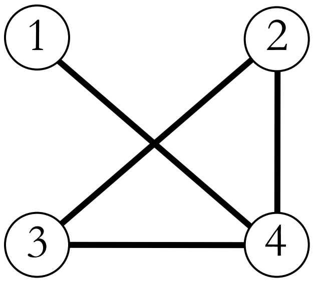
- For each edge (*v*, *w*) of *G*, add a vertex *vw* and the edges (*v*, *vw*) and (*w*, *vw*) to *G'*
- Furthermore, remove all vertices with no incident edges; such vertices would always have to go in a dominating set but are not needed in a vertex cover of *G*
  - We skip the discussion of this subtle part in the followings



[Recap] Vertex cover

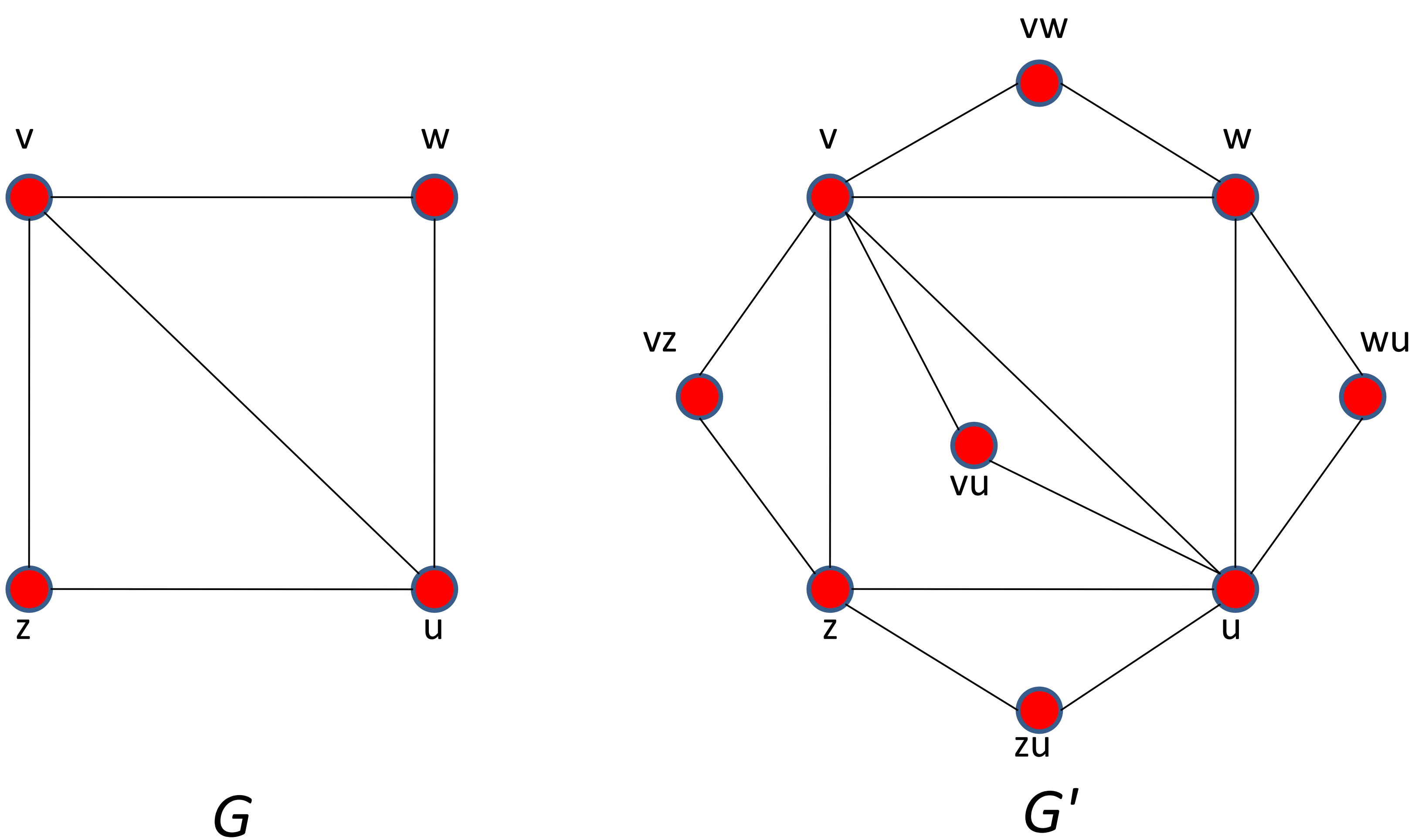
A vertex cover, C, is a set of vertices that covers all edges

- i.e., each edge is at least adjacent to some node in C



{2, 4}, {3, 4}, {1, 2, 3}
are vertex covers

Dominating Set: Graph Transformation Example



Dominating Set - (3) Correspondence

A dominating set of size *K* in *G'* ⇔ A vertex cover of size *K* in *G*

➔ Let D be a dominating set of size K in G'

- Case 1): D contains only vertices from G
  - Then, all new vertices have an edge to a vertex in D
  - D covers all edges
  - D is a valid vertex cover of G

Dominating Set - (3) Correspondence

A dominating set of size *K* in *G'* ⇔ A vertex cover of size *K* in *G*

➔ Let D be a dominating set of size K in G'

- Case 2): D contains some new vertices (vertex in the form of uv)

  (We show how to construct a vertex cover using only old vertices, otherwise we cannot obtain a vertex cover for G)

  For each new vertex uv, replace it by u (or v)

  If u ∈ D, this node is not needed

  Then the edge u-v in G will be covered

  After new edges are removed, it is a valid vertex cover of G (of size at most K)

Dominating Set - (3) Correspondence

A dominating set of size *k* in *G'* ⇔ A vertex cover of size *k* in *G*

🡐 Let C be a vertex cover of size *k* in G

For an old vertex, v ∈ G' :

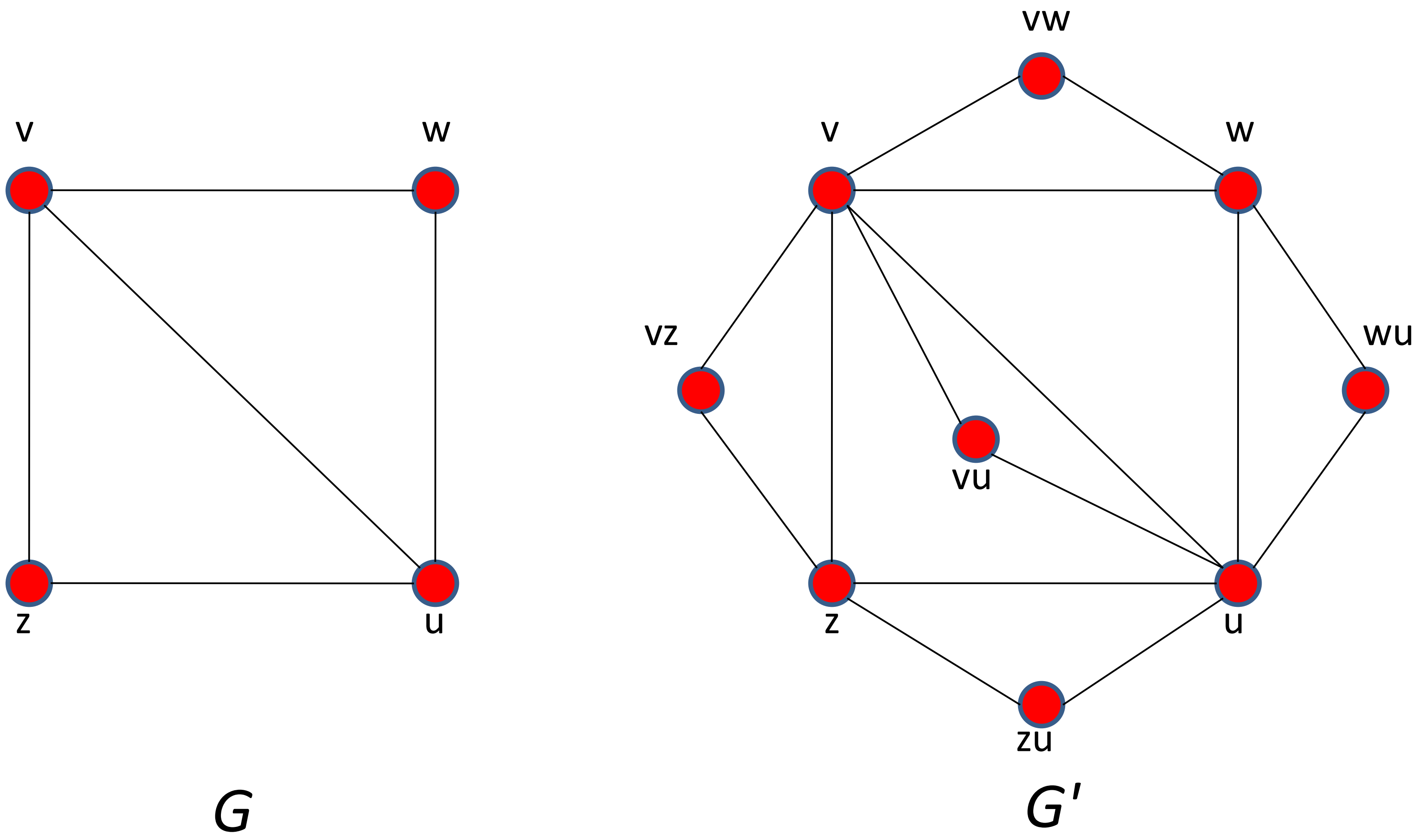
- By the definition of VC, all edges incident to v are covered
- v is also covered

For a new vertex, uv ∈ G' :

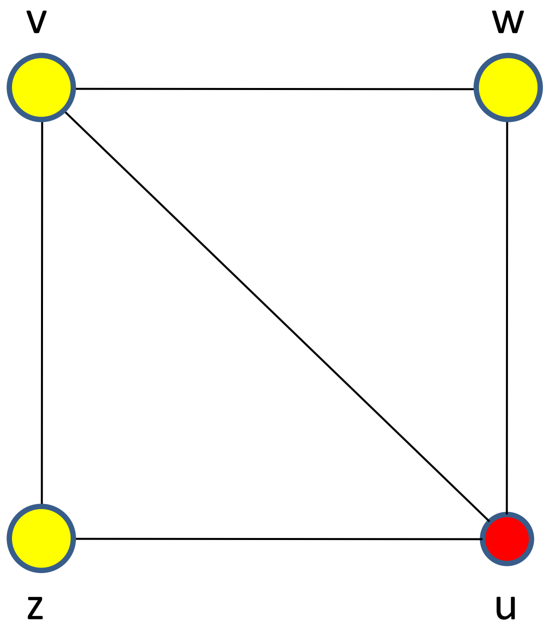
- Edge u-v must be covered, either u or v ∈ C
- This node will cover uv in G'

Thus, C is a valid dominating for G' (of size at most *k*)

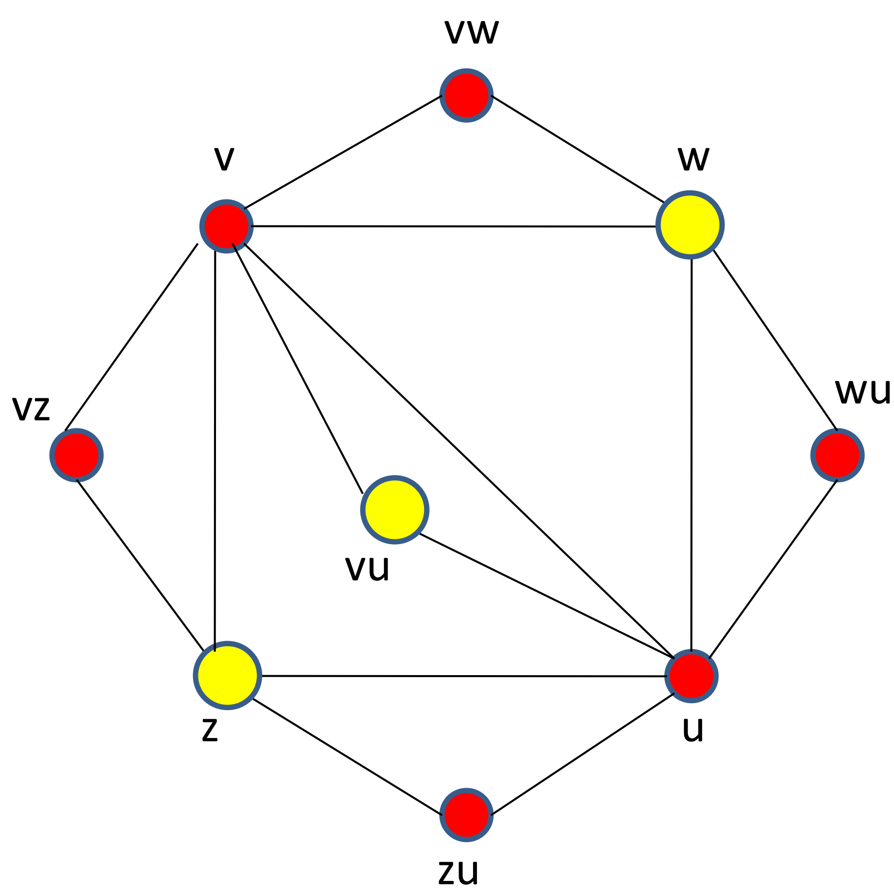
Dominating Set: Graph Transformation Example



Dominating Set - (3)
Correspondence



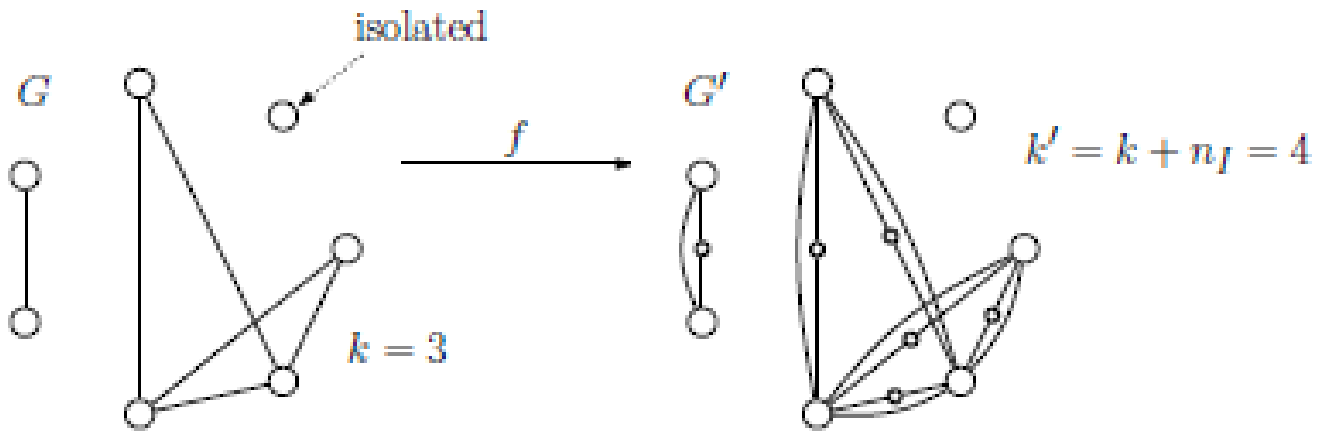
Vertex-cover *in G*



Dominating-set in *G′*

Vertex Cover to Dominating Set

VC $\leq_P$ DS



Dominating set reduction with k = 3 and one isolated vertex.

VC: “every edge is incident to a vertex in V′ ”.

DS: “every vertex is either in V′ or is adjacent to a vertex in V′ ”.

Translation must somehow map the notion of “incident” to “adjacent”

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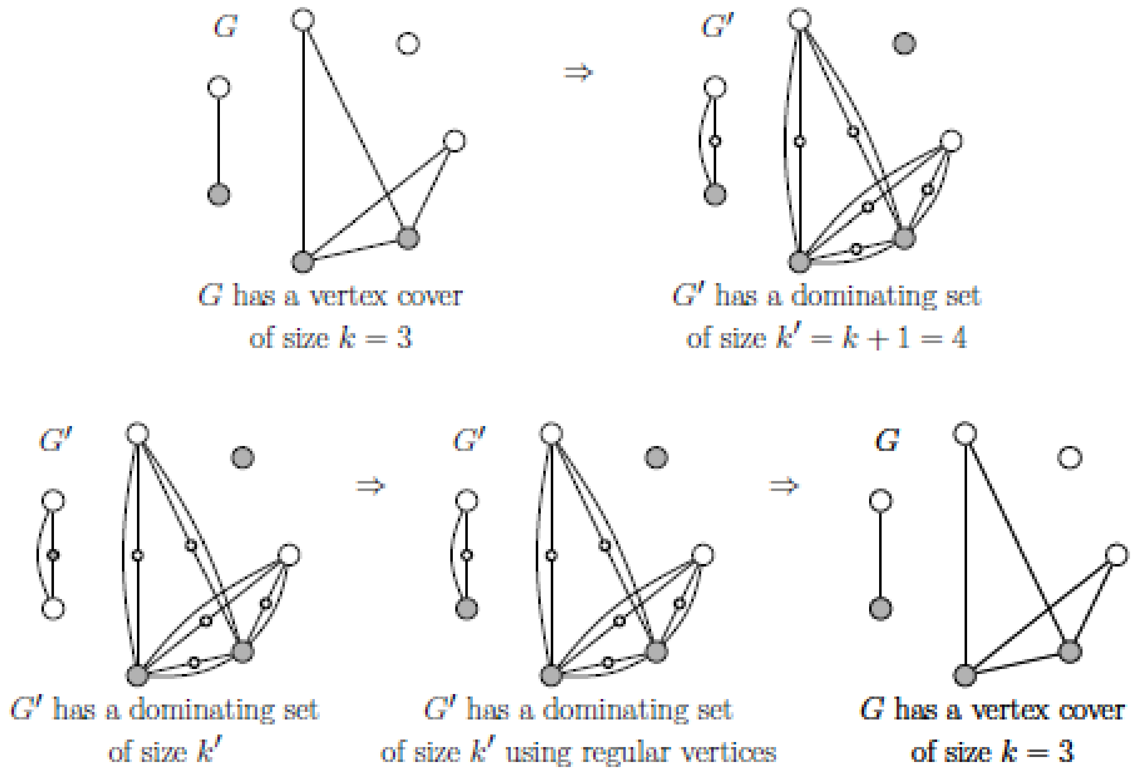
Correctness of the Reduction

We need to show that G has a vertex cover of size k if and only if G′ has a dominating set of size k′.



Correctness of the VC to DS reduction (where k = 3 and l = 1).

NP-Completeness

So far, we have seen:

1. 3-SAT to INDEPENDENT SET (IS)
2. IS to CLIQUE
3. IS to VERTEX COVER
4. VERTEX COVER to DOMINATING SET
5. 3-COLORING to CLIQUE COVER (not the same as CLIQUE)

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So your problem is NP-Complete? Now What?

Important: NP-Completeness is not a death sentence, but you need appropriate expectations/strategies

Some Useful Strategies

1. Brute-Force (for small input sizes)
2. Heuristics – Fast algorithms that are not always correct
3. Solve in exponential time but faster than brute-force search
4. Approximation Algorithms

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