Discrete Structures, Fall 2013, Homework 6

You must write the solutions to these problems legibly on your own paper, with the problems in sequential order, and with all sheets stapled together.

Prove each of the following statements true or false.

1. For any integer n, $n^2 + 5$ is not divisible by 4.

   Hint: Do this by contradiction. Use the Q-R theorem, but you don't need to use $d = 4$. A smaller divisor works. Each case the Q-R theorem gives you should lead to a contradiction.

2. The cube root of 5 is irrational.

   Hint: Do this like we did in class, by using a proof by contradiction. Assume the cube root is rational, and when you invoke the definition of rational, don't forget to state that the rational form a/b is in lowest terms, as this will be the fact you contradict later.

   Hint 2: Use our extra theorem from class: $\forall a, x \in \mathbb{Z}^+ \ \forall p \in \text{PRIMES} \ p \mid a^x \rightarrow p \mid a$.

3. For any integer a, if $6 \mid (3 - a)$, then $3 \nmid (a - 2)$.

   Hint: the $\nmid$ symbol means "does not divide." It is the negation of the divides predicate: $x \nmid y \Leftrightarrow \sim(x \mid y)$

Below are some extra midterm practice problems. You don't need to turn these in; they are not part of the homework.

1. For any prime number p, if $p > 3$ then p can be written as either $6q + 1$ or $6q + 5$ for some integer q.

   Hint: Use the QRT with $d = 6$, but but four cases lead to contradictions.

2. $\log_5(2)$ is irrational.

   Hint: Do this by contradiction. Work with the equation using the laws of logarithms and exponents (see the back inside cover of the book if you need a refresher) until you get an equation with one prime number raised to a power on each side of the equals sign. Then use the unique prime factorization theorem to show that there are no numbers that can make the equation true.

3. This problem comes straight from last year's first midterm:

   We say an number is a *triangular number* if and only if it can be written in the form $k(k+1)/2$ for some integer k. In other words,

   $$\text{Triangular}(x) \Leftrightarrow \exists k \in \mathbb{Z} \ x = k(k+1)/2.$$

   This definition comes from the fact that if x is a (positive) triangular number, then a group of x objects can be arranged in an equilateral triangle figure. Prove the following (true) statement:

   For any integer t, if t is a triangular number, then $8t + 1$ is a perfect square.

   Hint: Recall that the definition of perfect square is:

   $$\text{PerfectSquare}(x) \Leftrightarrow \exists k \in \mathbb{Z} \ x = k^2.$$