Discrete Structures, Fall 2014, Problem Set 6

You must write the solutions to these problems legibly on your own paper, with the problems in sequential order, and with all sheets stapled together.

Remember, an example may only be used to prove that an existential statement is true or a universal statement is false. Any example or counter-example must include specific values for the variables and enough algebra and justification to illustrate that the example proves what you are claiming it proves.

1. Prove $\forall n \in \mathbb{Z}^{\geq 2}\ \prod_{i=2}^{n} \left(1 - \frac{1}{i^2}\right) = \frac{n+1}{2n}$

2. Prove $\forall n \in \mathbb{Z}^{+}\ 11^n - 6$ is divisible by 5.

   Hint: $11 = 10 + 1$.

3. Suppose we define a sequence as follows:

   $$a_1 = 3;\ a_2 = -1;\ \text{and for all integers } i \geq 3,\ a_i = (a_{i-1})^2 + a_{i-2} + 1.$$

   Prove $\forall n \in \mathbb{Z}^{\geq 1}\ a_n$ is odd.

   Hint: Use strong induction. Your base cases are proving $P(1)$ and $P(2)$. Your inductive hypothesis will be "Suppose k is an arbitrary integer ≥ 2, and assume for all integers i, $1 \leq i \leq k$, that $P(i)$ is true." (You should rewrite this substituting in the real definition of P.)

4. Suppose we define a sequence as follows:

   $$b_0 = 2;\ b_1 = 7;\ \text{and for all integers } i \geq 2,\ b_i = 3b_{i-1} - 2b_{i-2}.$$

   Prove $\forall n \in \mathbb{Z}^{\geq 0}\ \exists q \in \mathbb{Z}\ b_n = 5q + 2$.

   Hint: Use strong induction.

   Hint: Define $P(n)$ to be "$\exists q \in \mathbb{Z}\ b_n = 5q + 2$". This proof is very similar to the previous one; follow the same basic idea (you again need two base cases, but your IH will be slightly different because the sequence subscripts start at 0, not 1).