Discrete Structures, Fall 2016, Homework 10 Practice & Solutions

You must write the solutions to these problems legibly on your own paper, with the problems in sequential order, and with all sheets stapled together.

1. Prove the following statement using an *element proof*:

   For any four sets A, B, C, and D, if $A \subseteq B$ and $C \subseteq D$, then $A \cap C \subseteq B \cap D$.

2. Prove the following statement using an *element proof*:

   For any three sets A, B, and C, if $A \subseteq (B - C)$, then $A \cap C = \emptyset$.

3. Prove the following statement using an *element proof*:

   For any three sets A, B, and C, if $A \subseteq B$ and $B \subseteq C$, then $A \times B \subseteq B \times C$.

4. Prove the following statement using an *algebraic proof*:

   For any two sets A and B, $((A^c \cup B^c) - A)^c = A$.

Solutions:

1. Prove the following statement using an *element proof*:

   For any four sets A, B, C, and D, if $A \subseteq B$ and $C \subseteq D$, then $A \cap C \subseteq B \cap D$.

   Proof:

   Let A, B, C, and D be arbitrary sets.

   Assume $A \subseteq B$ and $C \subseteq D$.

   [*Note: Our goal is to prove that $A \cap C \subseteq B \cap D$. Equivalently, we can rewrite this goal as $\forall x \in U\ (x \in A \cap C) \rightarrow (x \in B \cap D)$. We can prove this statement by letting x be an arbitrary element in U, assuming $x \in A \cap C$, and proving $x \in B \cap D$.*]

   Let x be an arbitrary element in U. Assume $x \in A \cap C$.

   [*Note: The line above could also be written as "Let x be an arbitrary element in $A \cap C$."*]

   $x \in A \cap C$ (from above)

   $x \in A \wedge x \in C$ (def of intersection)

   $x \in A$ (conjunctive simplification)

   $x \in C$ (conjunctive simplification)

   $x \in B$ (because $x \in A$ and $A \subseteq B$)

   $x \in D$ (because $x \in C$ and $C \subseteq D$)

   $x \in B \wedge x \in D$ (conjunctive addition)

   $x \in B \cap D$ (def of union)

   Because we have shown that if x is in $A \cap C$ then x is also in $B \cap D$, we can conclude (by the definition of subset) that $A \cap C \subseteq B \cap D$.

2. Prove the following statement using an *element proof*:

   For any three sets A, B, and C, if $A \subseteq (B - C)$, then $A \cap C = \emptyset$.

   Proof:

   Let A, B, and C be arbitrary sets.

   [*Note: Our goal is to prove that $A \cap C = \emptyset$. To show a set is equal to the empty set, we will do a proof by contradiction: assume the set is non-empty, and derive a contradiction.*]

   Assume by way of contradiction that $A \cap C \neq \emptyset$.

   Since $A \cap C$ is non-empty, there must exist an element in it: $\exists x \in U \ x \in A \cap C$.

   $x \in A \wedge x \in C$ (def of intersection)

   $x \in A$ (conjunctive simplification)

   $x \in C$ (conjunctive simplification)

   $x \in B - C$ (because $x \in A$, and $A \subseteq (B - C)$).

   $x \in B \wedge x \notin C$ (def of set difference)

   $x \notin C$ (conjunctive simplification)

   We now have a contradiction: $x \in C$ and $x \notin C$. Therefore, our assumption (that $A \cap C \neq \emptyset$) must have been incorrect, and we can say that $A \cap C = \emptyset$.

3. Prove the following statement using an *element proof*:

   For any three sets A, B, and C, if $A \subseteq B$ and $B \subseteq C$, then $A \times B \subseteq B \times C$.

   Proof:

   Let A, B, and C be arbitrary sets, and assume that $A \subseteq B$ and $B \subseteq C$.

   [*Note: We want to show $A \times B \subseteq B \times C$, so we'll do this with the standard method to prove a set is a subset of another set. We'll pick an arbitrary element in $A \times B$ and show that element must also be in $B \times C$.*]

   Let (x, y) be an arbitrary element in $A \times B$.

   $x \in A \wedge y \in B$ by definition of Cartesian product.

   $x \in A$ by conjunctive simplification.

   $y \in B$ by conjunctive simplification.

   $x \in B$ because $A \subseteq B$ and $x \in A$.

   $y \in C$ because $B \subseteq C$ and $y \in B$.

   $x \in B \wedge y \in C$ by conjunctive addition.

   $(x, y) \in B \times C$ by definition of Cartesian product.

   Because we have shown that if we choose an arbitrary element of $A \times B$, then that element must also be in $B \times C$, then we can conclude that $A \times B \subseteq B \times C$.

4. Prove the following statement using an *algebraic proof*:

   For any two sets A and B, $((A^c \cup B^c) - A)^c = A$.

   [*Note: We can do an algebraic proof here, because there are no initial assumptions about A or B (like one being a subset of the other or something similar). If there were any assumptions, we would need to do element proofs for subset in both directions to show equality.*]

   Proof:

$$\begin{aligned}
((A^c \cup B^c) - A)^c &= ((A^c \cup B^c) \cap A^c)^c && \text{def of set difference} \\
&= (A^c \cup B^c)^c \cup A && \text{DeMorgan's law} \\
&= (A \cap B) \cup A && \text{DeMorgan's law} \\
&= A && \text{absorption law}^*
\end{aligned}$$

   *Check the cheatsheet! Absorption is one of those rules that doesn't come up very often.