Discrete Structures, Fall 2016, Homework 9

You must write the solutions to these problems legibly on your own paper, with the problems in sequential order, and with all sheets stapled together.

Prove the following statements by *strong* induction. Make sure to follow the form from class: explicitly define $P(n)$, label the basis step, inductive step, where you define the inductive hypothesis, where you define what you want to prove, and where you use the inductive hypothesis.

1. Suppose we define a sequence as follows:

$$a_1 = 2;\ a_2 = 4;\ \text{and for all integers } i \geq 3,\ a_i = \frac{a_{i-2} \cdot a_{i-1}}{2}.$$

   Prove that every term in the sequence is even.

   Hint: Your base cases are proving $P(1)$ and $P(2)$. Your inductive hypothesis will begin "Suppose k is an arbitrary integer ≥ 2. Assume $P(1) \wedge P(2) \wedge P(3) \wedge \cdots \wedge P(k)$."

   Hint 2: Note that you are not proving that $a_i = \frac{a_{i-2} \cdot a_{i-1}}{2}$. You are given this fact; you know it is true; you do not need to prove it. You are proving that every term in the sequence is even.

2. Suppose we define a sequence as follows:

$$b_0 = 2;\ b_1 = 7;\ \text{and for all integers } i \geq 2,\ b_i = 3b_{i-1} - 2b_{i-2}.$$

   Prove $\forall n \in \mathbb{Z}^{\geq 0}\ \exists q \in \mathbb{Z}\ b_n = 5q + 2$.

   Hint: Define $P(n)$ to be "$\exists q \in \mathbb{Z}\ b_n = 5q + 2$". This proof is very similar to the previous one; follow the same basic idea (you again need two base cases, but your IH will be slightly different because the sequence subscripts start at 0, not 1).

   Hint 2: Do not be thrown off by the "$\exists q \in \mathbb{Z}\ b_n = 5q + 2$" part. After all, for question 1, you were proving every term in the sequence was even, which uses the definition of even that is equivalent to "$\exists q \in \mathbb{Z}\ a_n = 2q$." All we're doing in this proof is altering the formulas slightly.

3. Suppose we define a sequence as follows:

$$c_1 = 1;\ c_2 = 2;\ \text{and for all integers } i \geq 3,\ c_i = \begin{cases} 1 + c_{i/2} & \text{if } i \text{ is even} \\ 1 + c_{(i-1)/2} & \text{if } i \text{ is odd} \end{cases}$$

   (a) Calculate c_3, c_4, c_5, and c_6.

   (b) **Prove** $\forall n \in \mathbb{Z}^+\ c_n \leq n$.

   Hint: Again, you will need strong induction for this. You will need to split your inductive case (the part where you prove $P(k+1)$) into two sub-cases.