Discrete Structures, Spring 2016, Homework 8

You must write the solutions to these problems legibly on your own paper, with the problems in sequential order, and with all sheets stapled together.

Prove the following statements by induction (use strong induction only where appropriate). Make sure to follow the form from class: explicitly define $P(n)$, label the basis step(s), inductive step, where you define the inductive hypothesis, where you define what you want to prove and where you use the inductive hypothesis.

1. Prove $\forall n \in \mathbb{Z}^+ \; 11^n - 6$ is divisible by 5.

   Hint: $11 = 10 + 1$.

2. Prove $\forall n \in \mathbb{Z}^+ \; \sum_{i=1}^{n} (i \cdot i!) = (n+1)! - 1$

   Hint: Recall that $n! = n(n-1)(n-2)\cdots 2 \cdot 1$, with $0!$ defined to be 1. However, an alternate formula involving recursion is the following:

   $$n! = \begin{cases} 1 & \text{if } n = 0 \\ n \cdot (n-1)! & \text{otherwise} \end{cases}$$

   This recursive definition will be useful because there will be a part of the inductive step where you need to rewrite $(k+2)(k+1)!$ as something different.

3. Prove $\forall n \in \mathbb{Z}^{\geq 2} \; \prod_{i=2}^{n} \left(1 - \frac{1}{i^2}\right) = \frac{n+1}{2n}$

4. Suppose we define a sequence as follows:

   $$a_1 = 1; \; a_2 = 3; \text{ and for all integers } i \geq 3, \; a_i = a_{i-2} + 2a_{i-1}.$$

   Prove that every term in the sequence is odd.

   Hint: Use strong induction. Your base cases are proving $P(1)$ and $P(2)$. Your inductive hypothesis will be "Suppose k is an arbitrary integer ≥ 2, and assume for all integers i, $1 \leq i \leq k$, that $P(i)$ is true." (You should rewrite this substituting in your definition of $P(n)$.)

5. Suppose we define a sequence as follows:

   $$b_0 = 2; \; b_1 = 7; \text{ and for all integers } i \geq 2, \; b_i = 3b_{i-1} - 2b_{i-2}.$$

   Prove $\forall n \in \mathbb{Z}^{\geq 0} \; \exists q \in \mathbb{Z} \; b_n = 5q + 2$.

   Hint: Define $P(n)$ to be "$\exists q \in \mathbb{Z} \; b_n = 5q + 2$". This proof is very similar to the previous one; follow the same basic idea (you again need two base cases, but your IH will be slightly different because the sequence subscripts start at 0, not 1).

   Hint 2: Do not be thrown off by the "$\exists q \in \mathbb{Z} \; b_n = 5q + 2$" part. Think about what that math means intuitively. Write out the first few terms of the sequence. What's the pattern among the numbers — what do they all have in common?