Discrete Structures, Spring 2016, Homework 9

You must write the solutions to these problems legibly on your own paper, with the problems in sequential order, and with all sheets stapled together.

For each of the following, give a proof of the statement **if it is true**, or a counterexample **if the statement is false**. Remember, counterexamples must include specific values and enough work shown to demonstrate that they are actual counterexamples.

1. For all sets A, B, and C, if $A \cup C = B \cup C$, then $A = B$.

2. For all sets A, B, C, and D, if $A \subseteq (B \cup C)$ and $D \subseteq B^c$, then $A \cap D \subseteq C$.

3. For all sets A, B, and C, if $A \subseteq B$ and $B \subseteq C$, then $A \times B \subseteq B \times C$.

   Hint: Because we're proving a Cartesian product is a subset of another Cartesian product, after you begin with "Assume $A \subseteq B$ and $B \subseteq C$," you next write "Let (x, y) be an arbitrary element in $A \times B$." Normally this line would be "Let x be an arbitrary element in $A \times B$," but because we're dealing with Cartesian products, we need that second variable (y).

4. For all sets A and B, if $A \cap B = \emptyset$ then $A \times B = \emptyset$.

5. For all sets A, B, and C, if $B \cap C \subseteq A$, then $(C - A) \cap (B - A) = \emptyset$.

6. For all sets A and B, $A \cup (B - A) = A \cup B$.

7. One of the "normal" DeMorgan's laws for sets is $(A \cup B)^c = A^c \cap B^c$. There is a "generalized" version of this set equality, which is the following:

   $$\text{For any sets } A_1, A_2, \ldots, A_n,\ (A_1 \cup A_2 \cup \cdots \cup A_n)^c = A_1^c \cap A_2^c \cap \cdots \cap A_n^c.$$

   You will prove this via induction on n. Define $P(n)$ as

   $$(A_1 \cup A_2 \cup \cdots \cup A_n)^c = A_1^c \cap A_2^c \cap \cdots \cap A_n^c$$

   and then you will prove $\forall n \in \mathbb{Z}^{\geq 2}\ P(n)$.

   Do this via regular/weak induction. The "trick" to this proof is that you have to "break off" the end of the sequence of unions just like you "break off" the last item in a summation. In other words, when you get to the part of the inductive step where you are manipulating $(A_1 \cup A_2 \cup \cdots \cup A_k \cup A_{k+1})^c$, break that into two sets, one set $S = A_1 \cup A_2 \cup \cdots \cup A_k$ and the second set being just A_{k+1}. Then

   $$(A_1 \cup A_2 \cup \cdots \cup A_k \cup A_{k+1})^c = (S \cup A_{k+1})^c$$

   Hint: You are allowed to use the "normal" DeMorgan's laws (for two sets) when you do this proof.