Tail Recursion and Accumulators

Recursion

Should now be comfortable with recursion:

- No harder than using a loop (Maybe?)
- Often much easier than a loop
  - When processing a tree (e.g., evaluate an arithmetic expression)
  - Avoids mutation even for local variables
- Now:
  - How to reason about *efficiency* of recursion
  - The importance of *tail recursion*
  - Using an *accumulator* to achieve tail recursion
  - [No new language features here]

Call-stacks

While a program runs, there is a *call stack* of function calls that have started but not yet returned

- Calling a function **`f`** pushes an instance of **`f`** on the stack
- When a call to **`f`** to finishes, it is popped from the stack

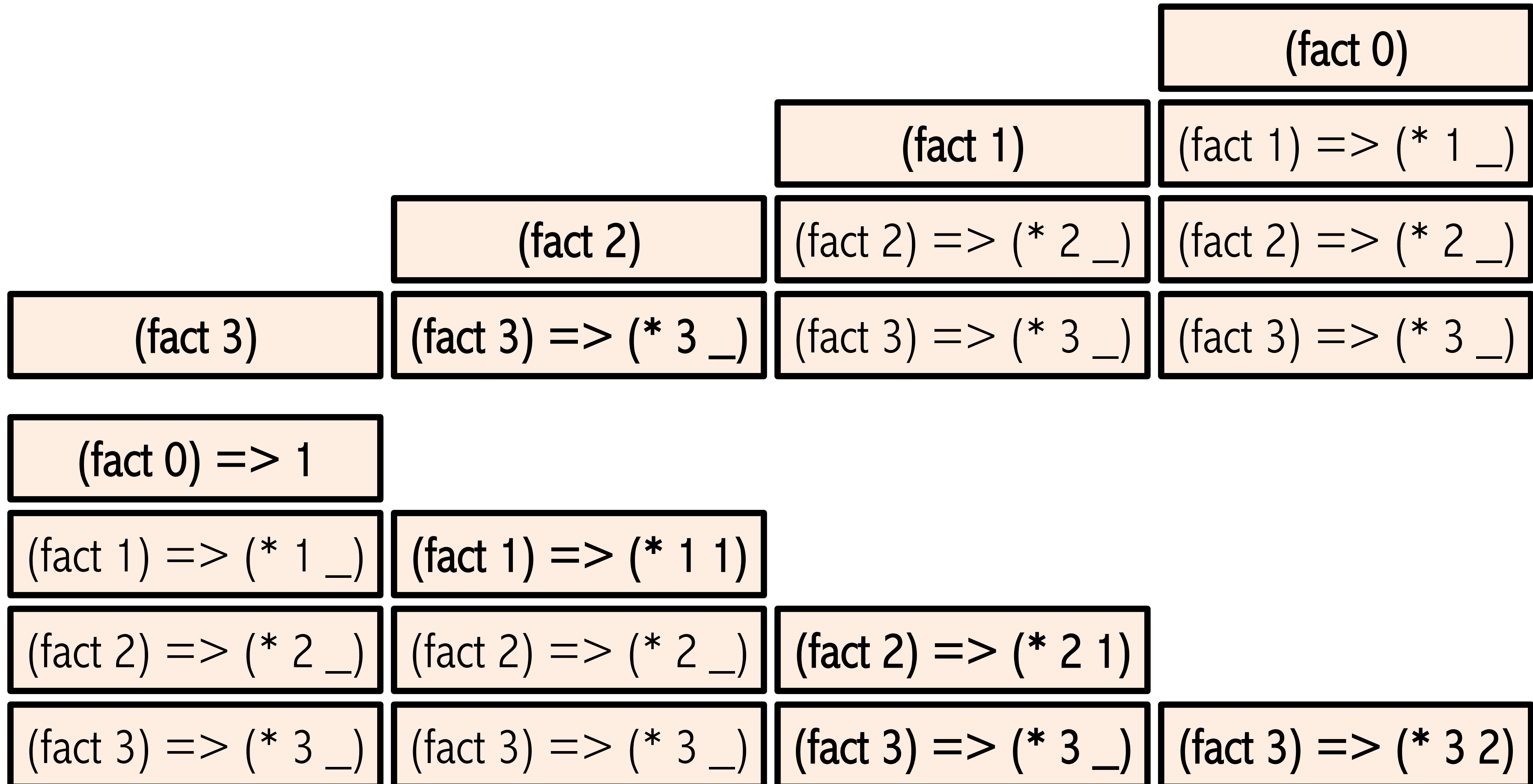
These *stack frames* store information such as

- the values of arguments and local variables
- information about "what is left to do" in the function (further computations to do with results from other function calls)

Due to recursion, multiple stack-frames may be calls to the same function

Example

```
(define (fact n)
  (if (= n 0) 1
    (* n (fact (- n 1)))))
```



What's being computed

```
(fact 3)
      => (* 3 (fact 2))
      => (* 3 (* 2 (fact 1)))
      => (* 3 (* 2 (* 1 (fact 0))))
      => (* 3 (* 2 (* 1 1)))
      => (* 3 (* 2 1))
      => (* 3 2)
      => 6
```

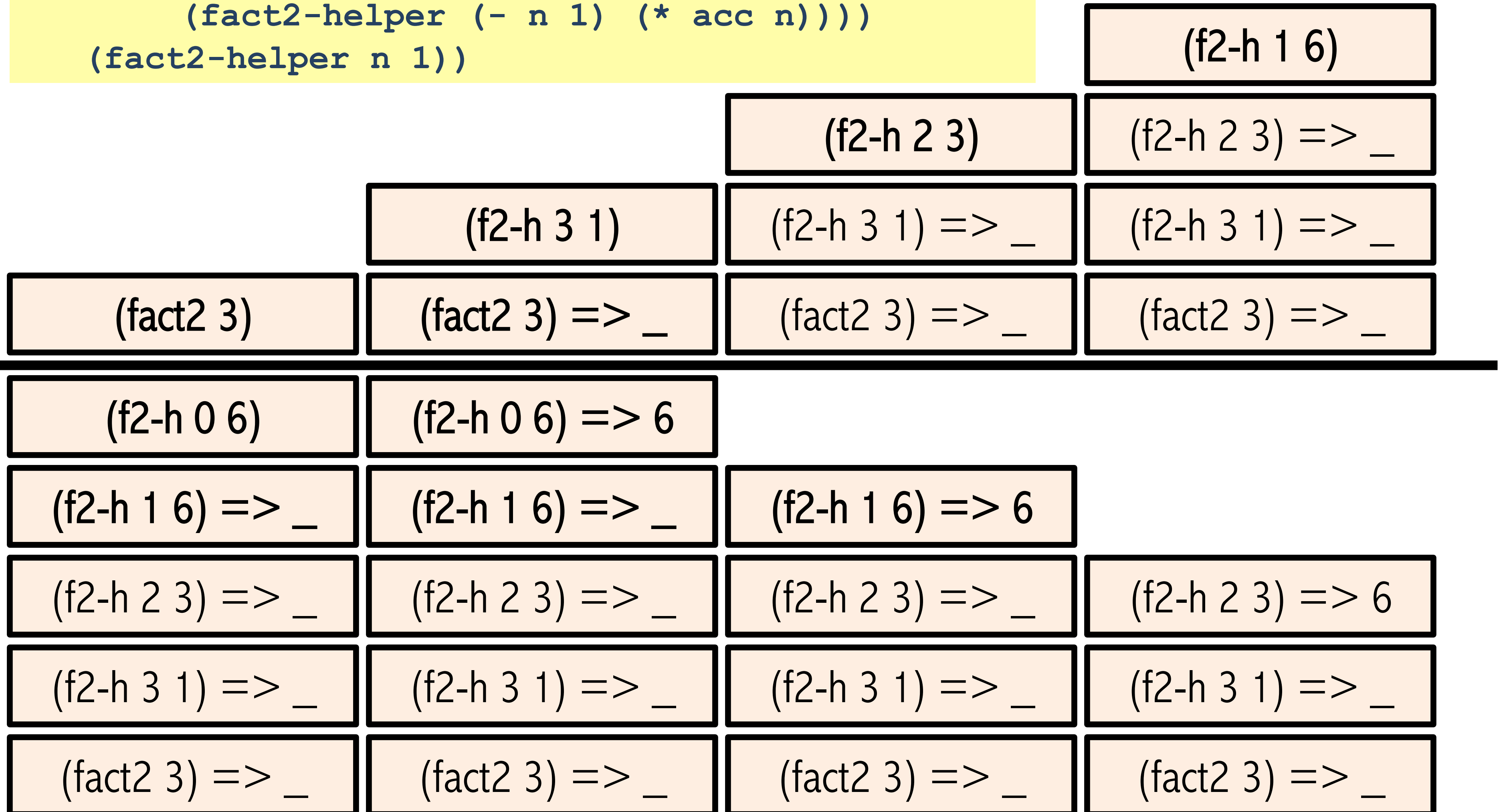
Example Revised

```
(define (fact2 n)

  (define (fact2-helper n acc)
     (if (= n 0) acc
        (fact2-helper (- n 1) (* acc n))))

  (fact2-helper n 1))
```

Still recursive, more complicated, but the result of recursive calls *is* the result for the caller (no remaining multiplication)

Example Revised

```
(define (fact2 n)
   (define (fact2-helper n acc)
    (if (= n 0) acc
        (fact2-helper (- n 1) (* acc n))))
   (fact2-helper n 1))
```



What's being computed

```
(fact2 3)
     => (fact2-helper 3 1)
     => (fact2-helper 2 3)
     => (fact2-helper 1 6)
     => (fact2-helper 0 6)
     => 6
```

An optimization

It is unnecessary to keep around a stack-frame just so it can get a callee's result and return it without any further evaluation

Racket recognizes these *tail calls* in the compiler and treats them differently:

- Pop the caller *before* the call, allowing callee to *reuse* the same stack space
- (Along with other optimizations,) as efficient as a loop

(Reasonable to assume all functional-language implementations do tail-call optimization)

includes Racket, Scheme, LISP, ML, Haskell, OCaml…

What really happens

```
(define (fact2 n)

  (define (fact2-helper n acc)
    (if (= n 0) acc
      (fact2-helper (- n 1) (* acc n))))

  (fact2-helper n 1))
```

(fact 3)	(f2-h 3 1)	(f2-h 2 3)	(f2-h 1 6)	(f2-h 0 6)

Moral

- Where reasonably elegant, feasible, and important, rewriting functions to be *tail-recursive* can be much more efficient
  - Tail-recursive: recursive calls are tail-calls
    - meaning all recursive calls must be the last thing the calling function does
    - no additional computation with the result of the callee
- There is also a methodology to guide this transformation:
  - Create a helper function that takes an *accumulator*
  - Old base case's return value becomes initial accumulator value
  - Final accumulator value becomes new base case return value

```
(define (fact n)
  (if (= n 0) 1
    (* n (fact (- n 1)))))
```

Old base case's return value becomes initial accumulator value.

```
(define (fact2 n)

  (define (fact2-helper n acc)
    (if (= n 0) acc
      (fact2-helper (- n 1) (* acc n))))

  (fact2-helper n 1))
```

Final accumulator value becomes new base case return value.

Another example

```
(define (sum1 lst)
  (if (null? lst) 0
    (+ (car lst) (sum1 (cdr lst)))))
```

```
(define (sum2 lst)

  (define (sum2-helper lst acc)
     (if (null? lst) acc
        (sum2-helper (cdr lst) (+ (car lst) acc))))

  (sum2-helper lst 0))
```

And another

```
(define (rev1 lst)
  (if (null? lst) '()
    (append (rev1 (cdr lst)) (list (car lst)))))
```

```
(define (rev2 lst)

  (define (rev2-helper lst acc)
    (if (null? lst) acc
      (rev2-helper (cdr lst) (cons (car lst) acc))))

  (rev2-helper lst '()))
```

Actually much better

```
(define (rev1 lst)     ; Bad version (non T-R)
  (if (null? lst) '()
     (append (rev1 (cdr lst)) (list (car lst)))))
```

- For **`fact`** and **`sum`**, tail-recursion is faster but both ways linear time
- The non-tail recursive **`rev`** is quadratic because each recursive call uses **`append`**, which must traverse the first list
  - And 1 + 2 + … + (length-1) is almost length * length / 2
  - Moral: beware **`append`**, especially if 1st argument is result of a recursive call
- **`cons`** is constant-time (and fast), so the accumulator version rocks

Tail-recursion == while loop with local variable

```
(define (fact2 n)
  (define (fact2-helper n acc)
    (if (= n 0) acc
      (fact2-helper (- n 1) (* acc n))))
   (fact2-helper n 1))
```

```
def fact2(n):
  acc = 1
  while n != 0:
    acc = acc * n
    n = n - 1
  return acc
```

Tail-recursion == while loop with local variable

```
(define (sum2 lst)
  (define (sum2-helper lst acc)
    (if (null? lst) acc
      (sum2-helper (cdr lst) (+ (car lst) acc))))
  (sum2-helper lst 0))
```

```
def sum2(lst):
  acc = 0
  while lst != []:
    acc = lst[0] + acc
    lst = lst[1:]
  return acc
```

Tail-recursion == while loop with local variable

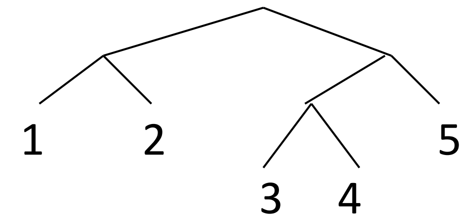
```
(define (rev2 lst)
  (define (rev2-helper lst acc)
    (if (null? lst) acc
      (rev2-helper (cdr lst) (cons (car lst) acc))))
  (rev2-helper lst '()))
```

```
def rev2(lst):
  acc = []
  while lst != []:
    acc = [lst[0]] + acc
    lst = lst[1:]
  return acc
```

Always tail-recursive?

There are certainly cases where recursive functions cannot be evaluated in a constant amount of space

Example: functions that process trees



- Lists can be used to represent trees: `'((1 2) ((3 4) 5))`

In these cases, the natural recursive approach is the way to go

- You could get one recursive call to be a tail call, but rarely worth the complication

Precise definition

If the result of **`(f x)`** is the "return value" for the enclosing function body, then **`(f x)`** is a tail call

i.e., don't have to do any more processing of (f x) to end function

Can define this notion more precisely…

- A *tail call* is a function call in *tail position*
- The single expression (ignoring nested defines) of the body of a function is in tail position.
- If **`(if test e1 e2)`** is in tail position, then **`e1`** and **`e2`** are in tail position (but **`test`** is not). (Similar for cond-expressions)
- If a let-expression is in tail position, then the single expression of the body of the **`let`** is in tail position (but no variable bindings are)
- Arguments to a function call are not in tail position
- …

Are these functions tail-recursive?

```
(define (get-nth lst n)
  (if (= n 0) (car lst)
    (get-nth (cdr lst) (- n 1))))

(define (good-max lst)
  (cond
    ((null? (cdr lst))
      (car lst))
    (#t
      (let ((max-of-cdr (good-max (cdr lst))))
        (if (> (car lst) max-of-cdr)
          (car lst) max-of-cdr)))))
```

Try these…

Write a tail-recursive max function (i.e., a function that returns the largest element in a list).

Write a tail-recursive Fibonacci sequence function (i.e., a function that returns the n'th number of the Fibonacci sequence).

```
(fib 1) => 1
(fib 2) => 1
(fib 3) => 2
(fib 4) => 3
(fib 5) => 5
```

In general, `(fib n) = (+ (fib (- n 1)) (fib (- n 2)))`

```scheme
(define (maxtr lst)
  (define (maxtr-helper lst max-so-far)
    (cond
      ((null? lst) max-so-far)
      ((> max-so-far (car lst))
          (maxtr-helper (cdr lst) max-so-far))
      (#t (maxtr-helper (cdr lst) (car lst)))))
  (maxtr-helper (cdr lst) (car lst)))
```

```
(define (fib-tr n)
  (define (fib-helper a b ctr)
    (if (= ctr n) a
        (fib-helper b (+ a b) (+ ctr 1))))
  (fib-helper 1 1 1))
```