Topics in topological combinatorics

Simplicial complexes, finite geometries, and the topology of circle-valued maps

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July 30, 2021

Ph.D. Thesis Defense
Carnegie Mellon University
Department of Mathematical Sciences
Program in Algorithms, Combinatorics, and Optimization

Thesis committee

I'd like to start by recognizing my thesis committee:

- Dr. Florian Frick (chair)
- Dr. Boris Bukh
- Dr. Gérard Cornuéjols
- Dr. Matthew Kahle (Ohio State University)

Outline

Topological combinatorics

Overview of dissertation

Vertex numbers of simplicial complexes

Topological combinatorics

What is combinatorics?

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(1) Start with a set of *vertices*.

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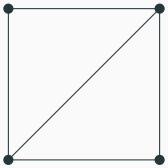
Example: A *simplicial complex* is constructed as follows:

(2) Add *edges* between some pairs of vertices.



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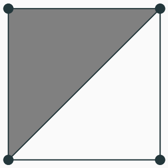

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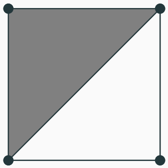

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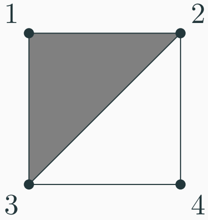
An (abstract) *simplicial complex* on a set S is a collection of subsets of S, such that if a subset f is in the collection, then all of f's subsets are also in the collection.



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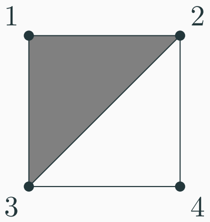

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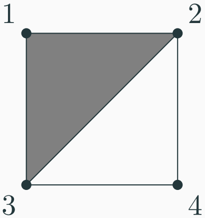

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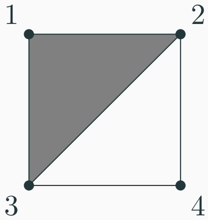

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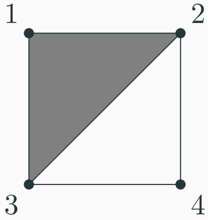

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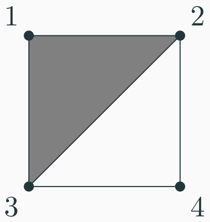

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The simplicial complex consists of a finite amount of information–just the sets listed to the right.

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Example: Simplicial complexes.

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Example: Count the simplicial complexes on $\{1, 2, 3\}$.

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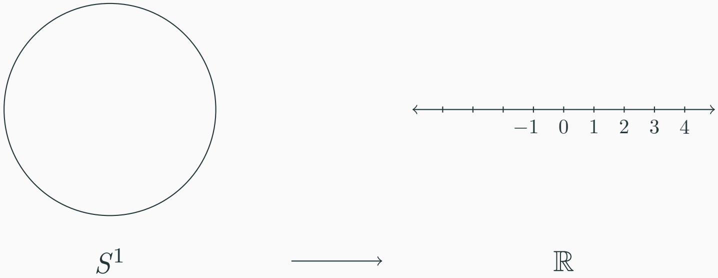
Example: The Borsuk-Ulam theorem in one dimension:



For any continuous function $f : S^1 \to \mathbb{R}$, there exists a point $x \in S^1$ with $f(-x) = f(x)$.

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Theorem (Borsuk-Ulam Theorem)
If $f : S^n \to \mathbb{R}^n$ is continuous, then there exists $x \in S^n$ with $f(-x) = f(x)$.

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- Solving combinatorial problems that arise in topological settings (chapter 2).
- Relating combinatorial structures to geometric ones (chapters 3, 4, 5).
- Applying the Borsuk-Ulam Theorem to problems not obviously about topology (chapter 6).

Overview of dissertation

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Joint work with Florian Frick.

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(a) *There is a simplicial complex X_n with $\pi_1(X_n) \cong \mathbb{Z}^n$ on $O(n)$ vertices.*

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Theorem

We have the following asymptotic results:

(a) *There is a simplicial complex X_n with $\pi_1(X_n) \cong \mathbb{Z}^n$ on $O(n)$ vertices.*

(b) *Every simplicial complex X_n with $\pi_1(X_n) \cong \mathbb{Z}^n$ has $\Omega(n^{3/4})$ vertices.*

Small cyclic simplicial complexes with fundamental group $\mathbb{Z}^n$ from projective planes

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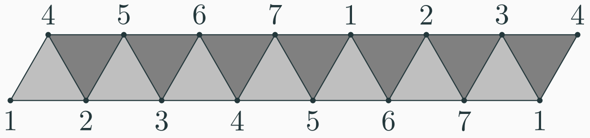

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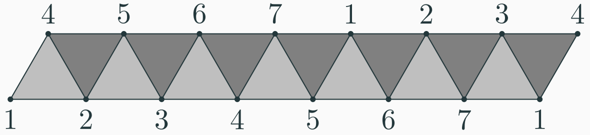

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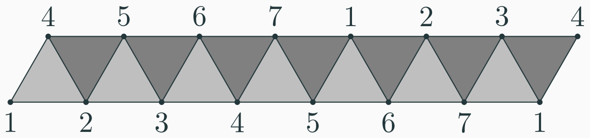

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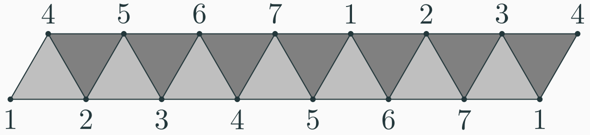

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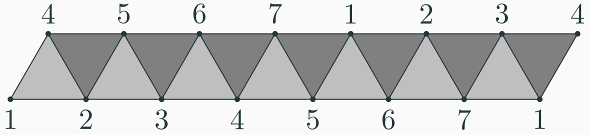

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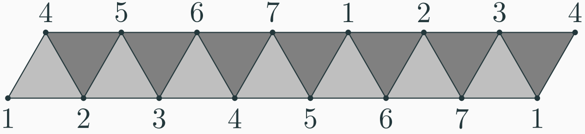

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Chapter 3: Simplicial complexes from projective planes

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We show a correspondence between colored k-configurations, Sidon sets, and linear codes.

Clean tangled clutters, simplices, and projective geometries

Joint work with Ahmad Abdi and Gérard Cornuéjols.

Submitted for publication.

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- If P is a simplex of dimension more than 3, then $\mathcal{C}$ has the Fano plane as a minor.

A new infinite class of ideal minimally non-packing clutters

Joint work with Ahmad Abdi and Gérard Cornuéjols.

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A clutter $\mathcal{C}$ is *ideal* if its set covering polyhedron is integral:

$$\{x \in \mathbb{R}_+^V : \sum_{v \in C} x_v \geq 1,\ C \in \mathcal{C}\}$$

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A nonlinear Lazarev-Lieb theorem: L^2-orthogonality via motion planning

Joint work with Florian Frick.

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Theorem (Hobby, Rice; 1965)
Let $f_1, \ldots, f_n \in L^1([0,1];\mathbb{R})$. *Then there exists* $h\colon [0,1] \to \{\pm 1\}$ *with at most* n *sign changes, such that for all* j,

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Unlike previous proofs, ours does not rely on the linearity of the integral, only the continuity.

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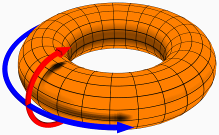

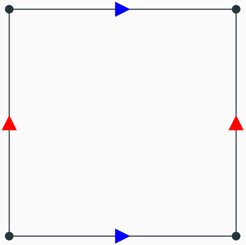

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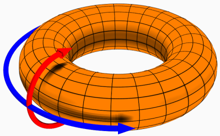

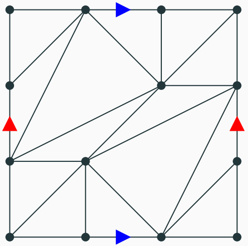

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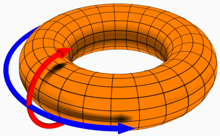

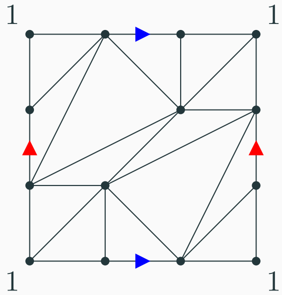

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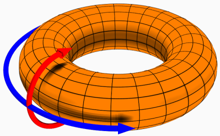

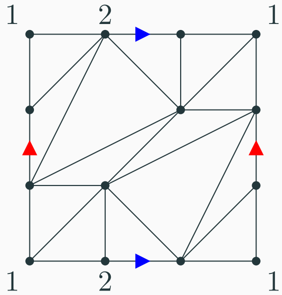

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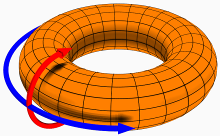

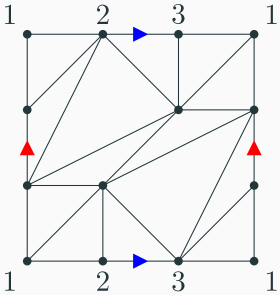

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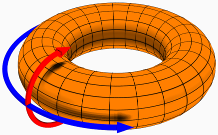

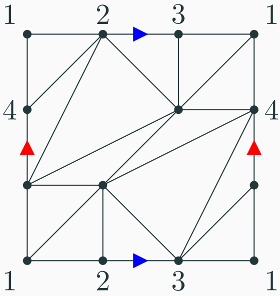

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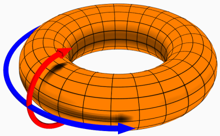

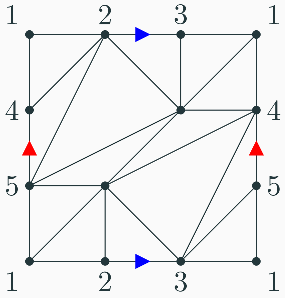

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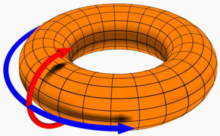

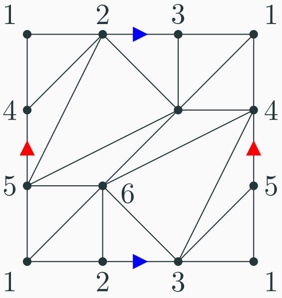

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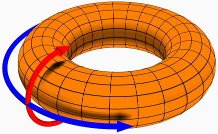

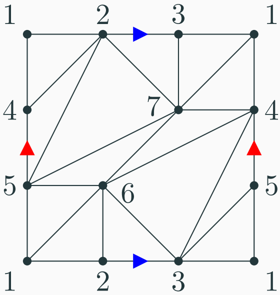

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Theorem (Arnoux, Marin; 1991)

Any triangulation of T^n has at least $\binom{n}{2}$ vertices.

The vertex number of fundamental group $\mathbb{Z}^n$.

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A related result:

Theorem (Kalai; 1983; Newman; 2018)
The number $T_d(G)$ of vertices required for a simplicial complex X with torsion part of $H_{d-1}(X)$ isomorphic to G satisfies:

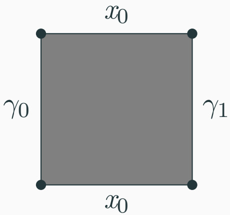
$$c_d(\log |G|)^{1/d} \leq T_d(G) \leq C_d(\log |G|)^{1/d}$$

The fundamental group $\pi_1(X; x_0)$

Given: a topological space X and a point $x_0 \in X$.

A *loop* at x_0 is a map $\gamma : [0,1] \to X$ with $\gamma(0) = \gamma(1) = x_0$.

A *homotopy* between loops γ_0, γ_1 at x_0 is a map $\delta : [0,1]^2 \to X$:



$$\delta(0,t) = \gamma_0(t)$$
$$\delta(1,t) = \gamma_1(t)$$
$$\delta(s,0) = x_0$$
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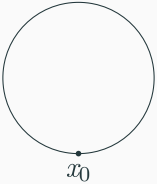
The elements of $\pi_1(X; x_0)$ are loops up to homotopy.

The group operation is concatenation.

The fundamental group: an example

Example: $\pi_1(S^1) \cong \mathbb{Z}$.



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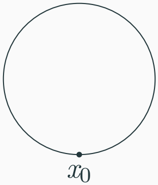

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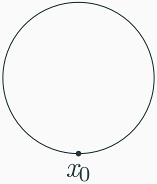

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(We omit the basepoint because $\pi_1(S^1; x_0)$ is independent of x_0, since S^1 is path-connected.)

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The fundamental group of a simplicial complex

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We define the fundamental group $\pi_1(X)$ as $\pi_1(|X|)$.

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Identify vertices and edges without changing $\pi_1(W_n)$.

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Vertices: u,

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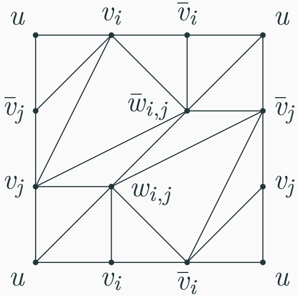

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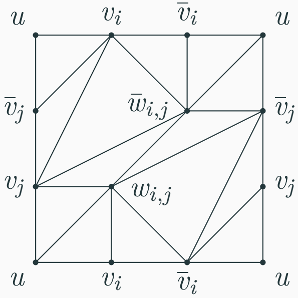

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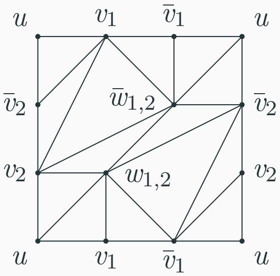

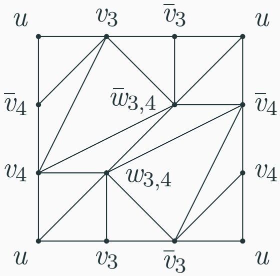

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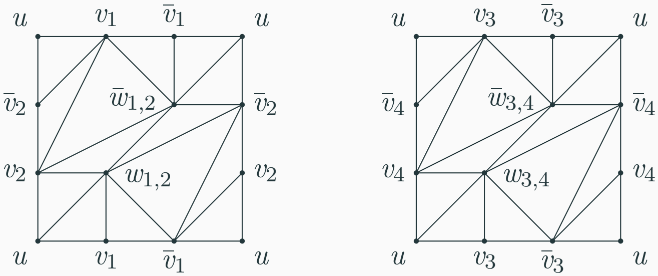

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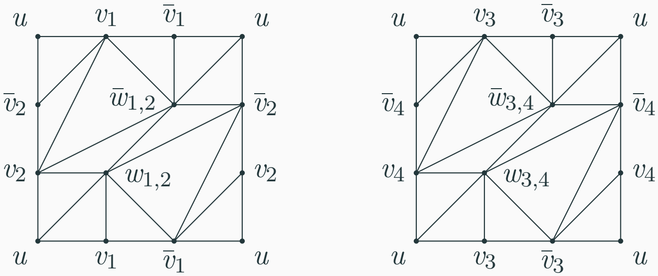

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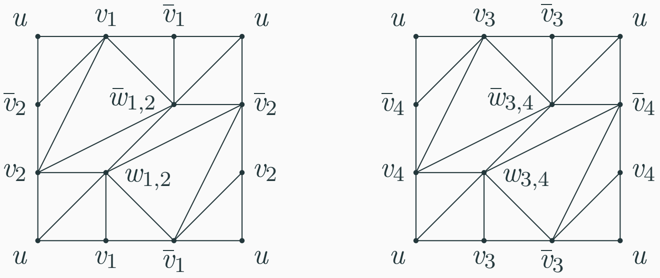

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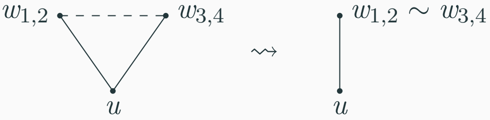

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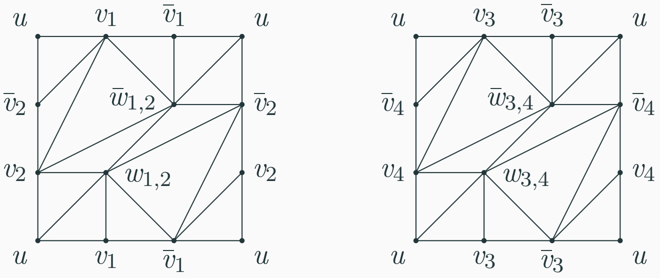

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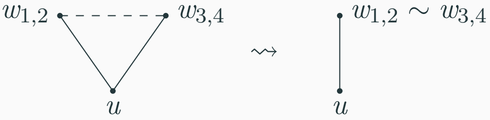

Use the fact that $w_{1,2}, w_{3,4}$ have no common neighbors.

Upper bound: Identifying many vertices in W_n.

We can similarly identify $w_{1,2}, w_{3,4}, w_{5,6}$ to a single vertex.

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For $n = 2k - 1$, a similar result follows by adding a dummy vertex to K_n, and proceeding as in the $n = 2k$ case.

Lower bound: Outline

Connection with group presentations

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Lemma

If X is a simplicial complex on k vertices with $\pi_1(X) \cong G$, then there exists a presentation $\langle S|R\rangle \cong G$ with $|S| \leq \binom{k}{2}$ and $|R| \leq \binom{k}{3}$.

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For example, $\langle g_1, \ldots, g_n | g_i g_j g_i^{-1} g_j^{-1},\ i < j\rangle \cong \mathbb{Z}^n$.

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So $k = \Omega(n^{2/3})$.

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Lower bound: Sylvester-Gallai theorems

Theorem (Sylvester, Melchior, Gallai; 1940)

Let S be a set of points in $\mathbb{R}^d$, such that for any distinct $x, y \in S$, there exists a third point $z \in S$ with x, y, z collinear.

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Then all points in S lie on a single line.

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Let S be a set of n points in $\mathbb{R}^d$, such that for any $x \in S$, for at least $\delta(n-1)$ of the remaining points $y \in S$, there exists a third point $z \in S$ with x, y, z collinear.

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Then all points in S lie in an affine subspace of dimension at most $12/\delta$.

Lower bound: Our Sylvester-Gallai variant

Let $V \subseteq \mathbb{R}^n$ be a finite set of points, and let E be a finite set of (not necessarily distinct) triples $\{u, v, w\}$ of distinct points $u, v, w \in V$ lying in a common 2-dimensional subspace of $\mathbb{R}^d$, so that (V, E) forms a 3-uniform hypergraph.

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Then for $\lambda > 0$, there exists an induced subhypergraph (V', E') of (V, E) with $|E| - |E'| < \lambda |V|$, and

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(V corresponds to S, E corresponds to R; take $\lambda = 24|S|/n$.)

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Apply Sylvester-Gallai to R to get $S' \subseteq S$:

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In (a), the exact number of vertices depends on parity:

For $n = 2k$, $k \neq 2, 3$, the complex X_n has $8k - 1$ vertices.

For $n = 2k - 1$, $k \neq 2, 3$, the complex X_n has $8k - 3$ vertices.

Further research

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Does a simplicial complex K with $\pi_1(K) \cong \mathbb{Z}^n$ require $\Omega(n)$ vertices?

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