Topics in topological combinatorics: Simplicial complexes, finite geometries, and the topology of circle-valued maps

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To my grandfather, David Superdock.

Contents

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Chapter 1

Introduction

Combinatorics is a loosely-organized area of mathematics having to do with counting and optimizing discrete mathematical structures. Topology is an area of mathematics having to do with deforming and distinguishing continuous mathematical structures. Topological combinatorics refers to the application of ideas from topology to problems in combinatorics.

This thesis consists of an introduction, followed by five papers on various topics in topological combinatorics. Each paper is given in its original form, as its own chapter, with co-authors and current publication status indicated at the beginning of the chapter. Here we give definitions and summarize the results of the later chapters, including directions for further research.

1.1 Simplicial complexes

Definition 1.1.1. A *simplicial complex* on a set S is a subset $K \subseteq \mathcal{P}(S)$, such that if $f \subseteq f'$ and $f' \in K$, then $f \in K$. The elements of K are called *faces*, and the elements of $\bigcup_{f \in K} f$ are called *vertices*. The *dimension* of a nonempty face $f \in K$ is $|f| - 1$. The faces of maximum dimension in K are called *facets*.

Definition 1.1.2. Let K be a simplicial complex on V. The *geometric realization* of K, denoted $|K|$, is the topological space in $\mathbb{R}^V$ given by

$$|K| = \left\{ \vec{x} \in \mathbb{R}^V : x_v \geq 0, \sum_{v \in V} x_v = 1, \text{ supp}(\vec{x}) \in K \right\}$$

(Here $\text{supp}(\vec{x}) = \{v \in V : x_v > 0\}$.)

Definition 1.1.3. Let K be a simplicial complex, let v be a vertex of K, and let $|v|$ be the corresponding point of $|K|$. Then the *fundamental group* of K with respect to basepoint v, denoted $\pi_1(K; v)$, is $\pi_1(|K|; |v|)$. (If $|K|$ is connected, then $\pi_1(K; v)$ is independent of v; in this case we use the notation $\pi_1(K)$.)

In Chapter 2, we prove the following results:

Theorem 2.1.1. *We have the following asymptotic results for vertex numbers of simplicial complexes with fundamental group* $\mathbb{Z}^n$*:*

(a) There is a simplicial complex X_n *with* $\pi_1(X_n) \cong \mathbb{Z}^n$ *on* $O(n)$ *vertices.*

(b) Every simplicial complex X_n *with* $\pi_1(X_n) \cong \mathbb{Z}^n$ *has* $\Omega(n^{3/4})$ *vertices.*

Theorem 2.2.8. *For* $n \in \mathbb{N}$ *with* $n \neq 2, 3$*, there exists:*

- *A simplicial complex* X_{2n} *with* $8n - 1$ *vertices,* $\pi_1(X_{2n}) \cong \mathbb{Z}^{2n}$*.*
- *A simplicial complex* X_{2n-1} *with* $8n - 3$ *vertices,* $\pi_1(X_{2n-1}) \cong \mathbb{Z}^{2n-1}$*.*

We pose the following question for further research:

Question 2.1.2. *Can we prove an analogue to Theorem 2.1.1 for fundamental group* $(\mathbb{Z}_k)^n$ *(perhaps restricting* k *to primes or prime powers)?*

1.2 Projective planes

Definition 1.2.1. A *projective plane* consists of sets P, L (whose elements are called "points" and "lines," respectively), and an incidence relation $R \subseteq P \times L$, satisfying the following conditions:

(1) For distinct $p_1, p_2 \in P$, there exists a unique $l \in L$ with $(p_1, l), (p_2, l) \in R$.

(2) For distinct $l_1, l_2 \in L$, there exists a unique $p \in P$ with $(p, l_1), (p, l_2) \in R$.

(3) There exist distinct $p_1, p_2, p_3, p_4 \in P$, such that for each $l \in L$, at most two of the four pairs (p_1, l), (p_2, l), (p_3, l), (p_4, l) are in R.

We say that the projective plane is *finite* if P and L are finite.

In a finite projective plane, there exists an integer q, called the *order* of the finite projective plane, such that each point is incident with exactly $q + 1$ lines, and each line is incident with exactly $q + 1$ points. Then it follows by a counting argument that the plane has exactly $q^2 + q + 1$ points and lines.

Definition 1.2.2. Let q be a prime power. Then $PG(2, \mathbb{F}_q)$ is a finite projective plane of order q, defined with reference to the vector space $\mathbb{F}_q^3$ over $\mathbb{F}_q$:

- Let P be the set of one-dimensional subspaces of $\mathbb{F}_q^3$.
- Let L be the set of two-dimensional subspaces of $\mathbb{F}_q^3$.
- Let $R \subseteq P \times L$ be the set of pairs (p, l) of subspaces p, l with $p \subseteq l$.

(Conditions (1) and (2) follow from the identity $\dim(U + V) + \dim(U \cap V) = \dim U + \dim V$, and for condition (3) we may take the four one-dimensional subspaces spanned by $(1, 0, 0), (0, 1, 0), (0, 0, 1), (1, 1, 1)$.)

In Chapter 3, we prove the following result:

Corollary 3.6.6. *Let q be a prime power. Then there exists a connected q-dimensional simplicial complex K with $\pi_1(K) \cong \mathbb{Z}^q$, such that:*

- *K has exactly $q^2 + q + 1$ vertices.*
- *K contains two copies of $PG(2, \mathbb{F}_q)$, each consisting of $q^2 + q + 1$ facets of K. These two copies fully describe K, in that these $2(q^2 + q + 1)$ facets are all of the facets of K, and every face of K is contained in a facet.*
- *K is cyclic; the cyclic group $\mathbb{Z}_{q^2+q+1}$ acts freely on K.*
- *K is 2-neighborly; that is, each pair of vertices in K form an edge.*

This result motivates the following conjecture:

Conjecture 3.1.2. *Suppose K is a simplicial complex on n vertices, such that $\pi_1(K) \cong \mathbb{Z}^k$, and such that K admits a free $\mathbb{Z}_n$-action. Then $n \geq k^2 + k + 1$, with equality attainable only for prime powers k.*

This conjecture implies that all cyclic planar difference sets have prime power order, a long-standing open problem in design theory.

1.3 Clutters

Definition 1.3.1. Let V be a finite set. A *clutter* is a family $\mathcal{C}$ of subsets of V, such that no set in $\mathcal{C}$ contains any other set in $\mathcal{C}$. (We refer to elements of V as *elements* of $\mathcal{C}$, and elements of $\mathcal{C}$ as *members* of $\mathcal{C}$.)

Definition 1.3.2. Let $\mathcal{C}$ be a clutter over ground set V. A *packing* of $\mathcal{C}$ is a set $P \subseteq \mathcal{C}$ of disjoint members. A *fractional packing* of $\mathcal{C}$ is a vector $\vec{x} \in [0,1]^{\mathcal{C}}$ with $\sum_{C \ni v} x_C \leq 1$ for all $v \in V$; its *value* is $\sum_{C \in \mathcal{C}} x_C$. The *packing number* $\nu(\mathcal{C})$ is the maximum cardinality of a packing of $\mathcal{C}$.

Definition 1.3.3. Let $\mathcal{C}$ be a clutter over ground set V. A *cover* of $\mathcal{C}$ is a subset $U \subseteq V$ with $U \cap C \neq \emptyset$ for all $C \in \mathcal{C}$. The *covering number* $\tau(\mathcal{C})$ is the minimum cardinality of a cover of $\mathcal{C}$.

We always have $\nu(\mathcal{C}) \leq \tau(\mathcal{C})$, by weak duality from linear programming.

Definition 1.3.4. Let $\mathcal{C}$ be a clutter. We say that $\mathcal{C}$ *packs* if $\nu(\mathcal{C}) = \tau(\mathcal{C})$. We say that $\mathcal{C}$ is *minimally non-packing* if $\mathcal{C}$ does not pack, but all proper minors of $\mathcal{C}$ pack.

Definition 1.3.5. A clutter $\mathcal{C}$ is *clean* if no minor of $\mathcal{C}$ is a delta or the blocker of an extended odd hole.

Definition 1.3.6. A clutter $\mathcal{C}$ is *tangled* if $\tau(\mathcal{C}) = 2$, and every element appears in a minimum cover.

Every clean tangled clutter has a fractional packing of value 2. The supports of all such packings of $\mathcal{C}$ form the *core* of $\mathcal{C}$, which has a simplified representation called the *setcore*.

Definition 1.3.7. A clutter $\mathcal{C}$ is *ideal* if its set covering polyhedron is integral:

$$\left\{x \in \mathbb{R}_+^V : \sum_{v \in C} x_v \geq 1,\ C \in \mathcal{C}\right\}$$

In Chapters 4 and 5, we prove the following results:

Theorem 4.1.5. *Let $\mathcal{C}$ be a clean tangled clutter of rank r. Then there exists a set $S \subseteq \{0,1\}^r$ such that the following statements hold:*

(i) $\operatorname{core}(\mathcal{C})$ *is a duplication of* $\operatorname{cuboid}(S)$*, and up to isomorphism, S is the unique set satisfying this property.*

(ii) There is a one-to-one correspondence between the fractional packings of value two in $\mathcal{C}$ and the different ways to express $\frac{1}{2} \cdot \mathbf{1}$ as a convex combination of the points in S.

(iii) $\operatorname{conv}(S)$ *is a full-dimensional polytope containing $\frac{1}{2} \cdot \mathbf{1}$ in its interior.*

Theorem 4.1.7. *Let $\mathcal{C}$ be a clean tangled clutter. Then* $\operatorname{conv}(\operatorname{setcore}(\mathcal{C}))$ *is a simplex if, and only if,* $\operatorname{setcore}(\mathcal{C})$ *is the cocycle space of a projective geometry.*

Theorem 4.1.8. *Let $\mathcal{C}$ be a clean tangled clutter where* $\operatorname{conv}(\operatorname{setcore}(\mathcal{C}))$ *is a simplex. If* $\operatorname{rank}(\mathcal{C}) > 3$*, then $\mathcal{C}$ has an $\mathbb{L}_7$ minor.*

Theorem 5.1.2. *Let $\mathcal{C}$ be a clutter, and let $G = G(\mathcal{C})$. Assume that*

- *G is bipartite and has exactly 3 connected components,*
- *the first connected component of G has two vertices $1, 2$ and an edge between them,*
- *the second connected component of G has two vertices $3, 4$ and an edge between them,*
- *the third connected component of G is a path on at least four edges, where the first edge is $\{5, 6\}$, the last edge is $\{5', 6'\}$, $5, 5'$ belong to the same part of the bipartition, and $6, 6'$ belong to the other part of the bipartition, and*
- *the minimal covers of $\mathcal{C}$ of cardinality different from two are precisely*

$$\{2,4,6\}, \{2,3,5\}, \{1,4,5'\}, \{1,3,6'\} \quad \textit{and} \quad \{3,5,6'\}, \{4,5',6\}.$$

Then $\mathcal{C}$ is an ideal minimally non-packing clutter.

We pose the following conjecture for further research:

Conjecture 4.6.1. *Every clean tangled clutter embeds a projective geometry over the two-element field.*

1.4 Circle-valued maps

In Chapter 6, we use the Borsuk-Ulam theorem to guarantee a simpler solution to a certain functional equation. Our main result is as follows:

Corollary 6.1.4. *Let* $f_1, \ldots, f_n \in L^1([0,1];\mathbb{C})$. *Then there exists* $h \in C^\infty([0,1]; S^1)$ *with* $\|h\|_{W^{1,1}} \leq 1 + 2\pi n$ *such that for all* j,

$$\int_0^1 f_j(x)h(x)dx = 0,$$

The $W^{1,1}$-norm is defined as follows:

$$\|h\|_{W^{1,1}} = \int_0^1 |h(x)|dx + \int_0^1 |h'(x)|dx.$$

As a result, we obtain bounds for the coindex of the space of smooth circle-valued functions with norm at most $1 + \pi n$:

Theorem 6.6.2. *For integer* $n \geq 1$ *let* Y_n *denote the space of* C^∞*-functions* $f\colon [0,1] \to S^1$ *with* $\|f\|_{W^{1,1}} \leq 1 + \pi n$. *Then*

$$n \leq \operatorname{coind} Y_n \leq 2n - 1.$$

We pose the following question for further research:

Problem 6.6.3. Determine the homotopy type of Y_n.

Chapter 2

Vertex numbers of simplicial complexes with free abelian fundamental group

Joint work with Florian Frick.

To be submitted for publication.

Abstract

We show that the minimum number of vertices of a simplicial complex with fundamental group $\mathbb{Z}^n$ is at most $O(n)$ and at least $\Omega(n^{3/4})$. For the upper bound, we use a result on orthogonal 1-factorizations of K_{2n}. For the lower bound, we use a fractional Sylvester-Gallai result.

We also prove that any group presentation $\langle S|R\rangle \cong \mathbb{Z}^n$ whose relations are of the form $g^a h^b i^c$ for $g, h, i \in S$ has at least $\Omega(n^{3/2})$ generators.

2.1 Introduction

Given a space X, a *vertex-minimal triangulation* of X is a simplicial complex homeomorphic to X using as few vertices as possible. Such triangulations are known for only a few manifolds [8, 17, 18], and upper and lower bounds differ significantly for many others, despite recent improvements such as [1]. For example, the n-dimensional torus can be triangulated on $2^{n+1} - 1$ vertices [16], but the best known lower bounds are quadratic in n; see [5].

The number of faces of a simplicial complex X can be bounded in terms of the Betti numbers of X [7] or in terms of the minimal number of generators of $\pi_1(X)$ [22]. The effect of relations of $\pi_1(X)$ on vertex numbers has been studied for cyclic torsion groups [15, 23] and for triangulations of manifolds with non-free fundamental group [25]. In this paper, we consider the minimal number of vertices of a simplicial complex with fundamental group $\mathbb{Z}^n$:

Theorem 2.1.1. *We have the following asymptotic results for vertex numbers of simplicial complexes with fundamental group $\mathbb{Z}^n$:*

(a) There is a simplicial complex X_n with $\pi_1(X_n) \cong \mathbb{Z}^n$ on $O(n)$ vertices.

(b) Every simplicial complex X_n with $\pi_1(X_n) \cong \mathbb{Z}^n$ has $\Omega(n^{3/4})$ vertices.

These results appear separately as Theorems 2.2.8 and 2.3.20; our precise upper bound depends on parity, but is asymptotically $4n$ in both cases:

Theorem 2.2.8. *For $n \in \mathbb{N}$ with $n \neq 2, 3$, there exists:*

- *A simplicial complex X_{2n} with $8n-1$ vertices, $\pi_1(X_{2n}) \cong \mathbb{Z}^{2n}$.*
- *A simplicial complex X_{2n-1} with $8n-3$ vertices, $\pi_1(X_{2n-1}) \cong \mathbb{Z}^{2n-1}$.*

To prove the $O(n)$ upper bound, we construct a complex W_n on n^2+n+1 vertices with fundamental group $\pi_1(W_n) \cong \mathbb{Z}^n$, and then perform identifications that preserve $\pi_1(W_n)$. The latter step uses a result on orthogonal 1-factorizations of the complete graph K_{2n}, which is implied by a result on Room squares; see [21, 14, 20].

To prove the $\Omega(n^{3/4})$ lower bound, we relate simplicial complexes to group presentations. Specifically, we define a *3-presentation* as a group presentation $\langle S|R\rangle$ whose relations are of the form $g^a h^b i^c$ for $g, h, i \in S$; this is a generalization of triangular presentations as studied in [3, 4, 2]. Then we show that simplicial complexes give rise to 3-presentations.

For any group presentation $\langle S|R\rangle \cong \mathbb{Z}^n$, it is known that $|R| \geq \binom{n}{2}$; this bound is sharp, by the presentation

$$\langle g_1, \ldots, g_n \,|\, g_i g_j g_i^{-1} g_j^{-1},\, i < j\rangle \cong \mathbb{Z}^n.$$

The bound $|R| \geq \binom{n}{2}$ already gives a $\Omega(n^{2/3})$ lower bound in Theorem 2.1.1(b). To strengthen this bound, we adapt the fractional Sylvester-Gallai results in [6, 11, 10], to show that any 3-presentation $\langle S|R\rangle \cong \mathbb{Z}^n$ has $|S| = \Omega(n^{3/2})$. This translates to a $\Omega(n^{3/4})$ lower bound in Theorem 2.1.1(b).

We conjecture that the $\Omega(n^{3/4})$ bound in Theorem 2.1.1(b) can be improved to $\Omega(n)$. We also pose the following question for further research:

Question 2.1.2. *Can we prove an analogue to Theorem 2.1.1 for fundamental group $(\mathbb{Z}_k)^n$ (perhaps restricting k to primes or prime powers)?*

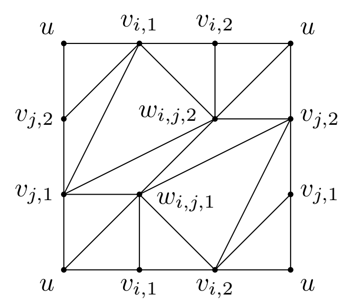
We will generally assume simplicial complexes X are connected, since if X is disconnected, and $\pi_1(X, v) \cong G$, then the component C of X containing v has fewer vertices than X, and $\pi_1(C, v) \cong G$. Under this assumption, $\pi_1(X, v)$ is independent of v, so we will write $\pi_1(X)$ instead.

2.2 Upper bound

In this section, we show that for all $n \in \mathbb{N}$, there exists a simplicial complex X_n on $O(n)$ vertices with fundamental group $\pi_1(X_n) \cong \mathbb{Z}^n$. We start by constructing a complex W_n on $n^2 + n + 1$ vertices with fundamental group $\pi_1(W_n) \cong \mathbb{Z}^n$, and then obtain X_n by identifying vertices and edges.

Definition 2.2.1. For $n \in \mathbb{N}$, the simplicial complex W_n is defined as follows:

- The vertex set consists of a vertex u, vertices $v_{i,k}$ for $i \in [n]$, $k \in [2]$, and vertices $w_{i,j,k}$ for $i, j \in [n]$, $i < j$, $k \in [2]$.
- The edge set includes edges $\{u, v_{i,1}\}, \{v_{i,1}, v_{i,2}\}, \{v_{i,2}, u\}$ for all $i \in [n]$.
- For each $i, j \in [n]$ with $i < j$, we include edges and triangles as in the following diagram. (The vertices and edges on the boundary are those defined above, and each planar region corresponds to a triangle.)



(This diagram also gives a vertex-minimal triangulation of the torus.)

Remark 2.2.2. *For all $n \in \mathbb{N}$, we have $\pi_1(W_n) \cong \mathbb{Z}^n$.*

Proof. Note that W_n is homeomorphic to a CW complex W_n' consisting of:

- A single 0-cell u.
- A 1-cell e_i from u to itself for each $i \in [n]$, corresponding to the edges $\{u, v_{i,1}\}, \{v_{i,1}, v_{i,2}\}, \{v_{i,2}, u\}$.
- A 2-cell $f_{i,j}$ attached along $e_i e_j e_i^{-1} e_j^{-1}$ for each $i, j \in [n]$, $i < j$, corresponding to the triangles in the diagram in Definition 2.2.1 for i, j. (By e_i^{-1} we denote attaching in the opposite direction along e_i.)

This gives a group presentation $\pi_1(W_n') \cong \langle S | R \rangle$, where

$$S = \{e_i : i \in [n]\}, \qquad R = \{e_i e_j e_i^{-1} e_j^{-1} : i, j \in [n],\ i < j\}.$$

But $\langle S | R \rangle \cong \mathbb{Z}^n$, so $\pi_1(W_n) \cong \pi_1(W_n') \cong \mathbb{Z}^n$, as desired. $\square$

We now establish the tools we need to perform identifications on W_n.

Definition 2.2.3. Let X be a simplicial complex with vertex $u \in V(X)$. We say that a set $S \subseteq V(X) \setminus \{u\}$ is a *spur* in (X, u) if the following properties hold:

(1) Each $v \in S$ is adjacent to u in X.

(2) No two distinct $v, v' \in S$ are adjacent in X.

(3) No two distinct $v, v' \in S$ have a common neighbor in X other than u.

Definition 2.2.4. Let X be a simplicial complex with vertex $u \in V(X)$. We say that two spurs S, S' in (X, u) are *compatible* if $S \cap S' = \emptyset$, and there is at most one edge $\{v, v'\}$ in X with $v \in S$, $v' \in S'$.

Lemma 2.2.5. *Let X be a simplicial complex with vertex $u \in V(X)$. If S is a spur in (X, u), then we can "collapse" S to obtain a simplicial complex, which we denote X/S, as follows:*

- *Identify all vertices $v \in S$ to a single new vertex w.*
- *Identify all edges $\{u, v\}$ for $v \in S$ to a single edge $\{u, w\}$.*

Moreover, $\pi_1(X/S) \cong \pi_1(X)$.

Proof. We may perform the identifications in the category of CW complexes, but we need to prove that the result is a simplicial complex. Since no adjacent vertices are identified, it remains to prove that no two distinct faces f, f' of X have the same vertex set in X/S, other than those explicitly identified.

If f, f' are distinct faces of X with the same vertex set in X/S, then there exist $v, v' \in S$ with $v \in f$, $v' \in f'$. If $f, f' \subseteq \{u\} \cup S$, then f, f' are either $\{v\}, \{v'\}$ or $\{u, v\}, \{u, v'\}$, and are explicitly identified. Hence we may assume that f, f' both contain a vertex $x \notin \{u\} \cup S$. But then x is a common neighbor of v, v', a contradiction. Hence X/S is a simplicial complex.

Now let A be the subcomplex of X with vertices u, S and edges $\{u, v\}$ for all $v \in S$, and let B be the subcomplex of X/S with vertices u, w and edge $\{u, w\}$. Consider the quotients X/A, $(X/S)/B$ in the category of CW complexes, and note that $X/A \cong (X/S)/B$. Since A, B are contractible, we have homotopy equivalences $X/A \simeq X$ and $(X/S)/B \simeq X/S$ (see [13], Proposition 0.17). By transitivity, we have $X \simeq X/S$, so $\pi_1(X) \cong \pi_1(X/S)$ as desired. □

Lemma 2.2.6. *Let X be a simplicial complex with vertex $u \in V(X)$.*

(a) If S, S' are compatible spurs in (X, u), then S' is a spur in $(X/S, u)$.

(b) If S, S', S'' are pairwise compatible spurs in (X, u), then S', S'' are compatible spurs in $(X/S, u)$.

Proof. For (a), conditions (1), (2) in Definition 2.2.3 hold since S, S' are disjoint, and condition (3) holds since S, S' have at most one edge between them.

For (b), S', S'' are spurs in $(X/S, u)$ by (a), and compatibility follows from the fact that collapsing S does not affect vertices or edges among $S' \cup S''$. □

Lemma 2.2.7. *For $n \in \mathbb{N}$ with $n \neq 2, 3$, the vertices $w_{i,j,k}$ of W_{2n} (or W_{2n-1}) can be partitioned into $4n-2$ pairwise compatible spurs in (W_{2n}, u) (or (W_{2n-1}, u)).*

Proof. A *1-factorization* of the complete graph on vertex set $[2n]$ is a partition of its edges into perfect matchings. An *orthogonal pair* of 1-factorizations is a pair of 1-factorizations, such that no two edges appear in the same matching in both factorizations. By [21] (see also [14, 20]), such a pair (F_1, F_2) exists for all $n \in \mathbb{N}$ with $n \neq 2, 3$.

Then for each matching $M \in F_k$, we construct a spur S_M in (W_{2n}, u):

$$S_M = \{w_{i,j,k} : \{i, j\} \in M\}$$

To see that S_M is a spur, note that the neighbors of $w_{i,j,k}$ in W_{2n} are u, and some vertices of the form $v_{i,k'}, v_{j,k'}, w_{i,j,k'}$ for $k' \in [2]$, so conditions (1) and (2) hold. Since M is a matching, condition (3) holds.

Then the S_M are disjoint, and the only edges between vertices in $S_M, S_{M'}$ for distinct M, M' are the edges $\{w_{i,j,1}, w_{i,j,2}\}$, which arise for $M \in F_1$, $M' \in F_2$ with $\{i, j\} \in M$, $\{i, j\} \in M'$. Then the orthogonality of (F_1, F_2) implies that the S_M are pairwise compatible, and there are $2(2n-1) = 4n-2$ such S_M.

Viewing W_{2n-1} as an induced subcomplex of W_{2n}, the S_M remain pairwise compatible spurs in (W_{2n-1}, u), upon deleting the missing vertices. □

Our promised upper bound follows:

Theorem 2.2.8. *For $n \in \mathbb{N}$ with $n \neq 2, 3$, there exists:*

- *A simplicial complex X_{2n} with $8n-1$ vertices, $\pi_1(X_{2n}) \cong \mathbb{Z}^{2n}$.*
- *A simplicial complex X_{2n-1} with $8n-3$ vertices, $\pi_1(X_{2n-1}) \cong \mathbb{Z}^{2n-1}$.*

Proof. Starting with W_{2n} or W_{2n-1}, apply Lemma 2.2.7 to obtain $4n-2$ pairwise compatible spurs. Collapse these spurs, one by one, via Lemma 2.2.5, to obtain X_{2n} or X_{2n-1} with $\pi_1(X_{2n}) \cong \mathbb{Z}^{2n}$, $\pi_1(X_{2n-1}) \cong \mathbb{Z}^{2n-1}$; note that Lemma 2.2.6 guarantees that the remaining spurs remain compatible. The remaining vertices are u, the $v_{i,k}$, and one vertex for each spur, so:

- The number of vertices in X_{2n} is $1 + 4n + (4n-2) = 8n-1$.
- The number of vertices in X_{2n-1} is $1 + 2(2n-1) + (4n-2) = 8n-3$.

This completes the proof. □

2.3 Lower bound

In this section, we show that a simplicial complex X with fundamental group $\pi_1(X) \cong \mathbb{Z}^n$ has $\Omega(n^{3/4})$ vertices. We begin by relating simplicial complexes to group presentations:

Definition 2.3.1. Given a group G, a *3-presentation* of G is a group presentation $\langle S|R\rangle \cong G$ where each relation in R is one of the following:

- $\langle\rangle$ (the empty word).
- g^a, where $g \in S$ and $a \in \mathbb{Z}$.
- $g^a h^b$, where $g, h \in S$ and $a, b \in \mathbb{Z}$.
- $g^a h^b i^c$, where $g, h, i \in S$ and $a, b, c \in \mathbb{Z}$.

We say such a word w is in *normal form* if the generators used are all distinct, and $a, b, c \neq 0$. To "write $r \in R$ in normal form" means to find w in normal form such that r and w are conjugates in $\langle S\rangle$; in this case we write $r \rightsquigarrow w$.

For example, we describe a 3-presentation $\langle S|R\rangle \cong \mathbb{Z}^n$, derived from the presentation for $\mathbb{Z}^n$ given in the introduction:

$$S = \{g_i : i \in [n]\} \cup \{h_{i,j} : i, j \in [n],\, i < j\}$$
$$R = \{g_i g_j h_{i,j},\, g_j g_i h_{i,j} : i, j \in [n],\, i < j\}$$

We will use the phrase, "Let $\phi\colon \langle S|R\rangle \cong G$ be a 3-presentation," to mean, "Let $\langle S|R\rangle \cong G$ be a 3-presentation, and fix an isomorphism $\phi\colon \langle S|R\rangle \to G$."

Remark 2.3.2. *Let $\langle S|R\rangle \cong G$ be a 3-presentation. Then any relation $r \in R$ can be written uniquely in normal form, up to the following conjugacies:*

- *$g^a h^b, h^b g^a$ are conjugates in $\langle S\rangle$.*
- *$g^a h^b i^c, h^b i^c g^a, i^c g^a h^b$ are conjugates in $\langle S\rangle$.*

Proof. If r is not in normal form, we can apply one of the following steps:

- If r has a zero exponent or identical adjacent generators, rewrite r with fewer generators. (For example, $g^0 hi$ becomes hi; $g^1 g^2 h$ becomes $g^3 h$.)
- If $r = g^a h^b g^c$, then replace r with its conjugate $g^{a+c} h^b$.

Each such step reduces k in $r = \prod_{i=1}^{k} g_i^{a_i}$, so this process terminates. Uniqueness follows from considering the conjugates of reduced words w. □

Lemma 2.3.3. *If X is a simplicial complex on k vertices with fundamental group $\pi_1(X) \cong G$, then there exists a 3-presentation $\langle S|R\rangle \cong G$ with $|S| \leq \binom{k}{2}$ and $|R| \leq \binom{k}{3}$.*

Proof. Assume X is connected, otherwise reduce to the component of X containing the basepoint. Then the 1-skeleton of X is a connected graph; choose a spanning tree T of this graph. View X as a CW complex and T as a contractible subcomplex of X, and consider the quotient complex X/T, which is homotopy equivalent to X (see [13], Proposition 0.17), so $\pi_1(X/T) \cong G$.

Now X/T has a single 0-cell, and hence can be viewed as the presentation complex of some group presentation $\langle S|R\rangle \cong G$ upon choosing a direction for each 1-cell. The 1-cells correspond bijectively to the generators, and arise from distinct edges of X, so $|S| \leq \binom{k}{2}$. The 2-cells correspond bijectively to the relations, and arise from distinct triangles of X, so $|R| \leq \binom{k}{3}$.

Moreover, each $r \in R$ is of one of the following forms, depending on how many edges of the corresponding triangle in X lie in T:

- g^a, where $g \in S$ are distinct and $a \in \{\pm 1\}$.
- $g^a h^b$, where $g, h \in S$ are distinct and $a, b \in \{\pm 1\}$.
- $g^a h^b i^c$, where $g, h, i \in S$ are distinct and $a, b, c \in \{\pm 1\}$.

In particular, $\langle S|R\rangle$ is a 3-presentation, which completes the proof. □

Hence we may turn our attention to proving lower bounds on $|S|$ and $|R|$ for 3-presentations $\langle S|R\rangle$ of given groups. We will use the concept of deficiency:

Definition 2.3.4. The *deficiency* of a group presentation $P = \langle S|R\rangle$ is def $P = |S| - |R|$. The *deficiency*, def G, of a group G is the maximum of def P over all presentations P of G.

Then we have an inequality in group homology due to Epstein [12]:

$$\text{def}\, G \leq \text{rank}\, H_1(G;\mathbb{Z}) - s(H_2(G;\mathbb{Z})),$$

where $s(H_2(G;\mathbb{Z}))$ is the minimum number of generators of $H_2(G;\mathbb{Z})$. (For a related inequality in terms of free resolutions, see [26].) In particular, when G is $\mathbb{Z}^n$, we obtain def $\mathbb{Z}^n \leq n - \binom{n}{2}$. Thus we have several constraints on the size of a presentation of $\mathbb{Z}^n$; if $\langle S|R\rangle \cong \mathbb{Z}^n$, then

- $|S| \geq n$.
- $|R| - |S| \geq \binom{n}{2} - n$.
- $|R| \geq \binom{n}{2}$ (by adding the previous two inequalities).

For the presentation of $\mathbb{Z}^n$ described in the introduction, we have equality in all three of these bounds. Hence def $\mathbb{Z}^n = n - \binom{n}{2}$.

Lemma 2.3.3 allows us to translate these bounds into lower bounds for the number of vertices in a simplicial complex X with fundamental group $\pi_1(X) \cong \mathbb{Z}^n$. The first inequality above gives a bound of $\Omega(n^{1/2})$, and the third gives a stronger bound of $\Omega(n^{2/3})$. We present the latter bound in more detail:

Remark 2.3.5. *A simplicial complex X with fundamental group $\pi_1(X) \cong \mathbb{Z}^n$ has at least $\Omega(n^{2/3})$ vertices.*

Proof. Let $f(n)$ be the minimum number of vertices in a simplicial complex X_n with fundamental group $\pi_1(X_n) \cong \mathbb{Z}^n$. By Lemma 2.3.3, for each n we obtain a 3-presentation $\langle S_n|R_n\rangle \cong \mathbb{Z}^n$ with $|R_n| \leq \binom{f(n)}{3}$. But $|R_n| \geq \binom{n}{2}$, so $\binom{f(n)}{3} \geq \binom{n}{2}$, hence $f(n) = \Omega(n^{2/3})$. □

Up to now, we have considered bounds on the size of arbitrary presentations of $\mathbb{Z}^n$. Now we turn to proving that for 3-presentations $\langle S|R\rangle \cong \mathbb{Z}^n$, we have a stronger bound $|S| = \Omega(n^{3/2})$. First we introduce a notion of dimension:

Definition 2.3.6. Let $\phi\colon \langle S|R\rangle \cong \mathbb{Z}^n$ be a 3-presentation. Then the *dimension* of a subset $S' \subseteq S$, denoted dim S', is

$$\dim(\text{span}\{\phi(g) : g \in S'\}),$$

where we view each $\phi(g)$ as a vector in $\mathbb{R}^n \supseteq \mathbb{Z}^n$.

For $r \in R$, let $r \rightsquigarrow \prod_i g_i^{a_i}$ by Remark 2.3.2. The *dimension* of r is the dimension of the subset $\{g_i\} \subseteq S$. (Note that the set $\{g_i\}$ is independent of the choice of normal form, so this definition is valid.)

Note that for a relation r with $r \rightsquigarrow \prod_{i=1}^k g_i^{a_i}$, we have $\sum_{i=1}^k a_i\phi(g_i) = 0$, a linear dependence among the $\phi(g_i)$. It follows that $\dim r < k$. In particular, all relations of a 3-presentation $\langle S|R\rangle \cong \mathbb{Z}^n$ have dimension at most two.

Our next goal is to show that for 3-presentations $\langle S|R\rangle \cong \mathbb{Z}^n$ with $|S|$ minimal, all nonempty relations have dimension exactly two. To do this, we use Tietze transformations ([27]; see also [19]):

Remark 2.3.7 (Tietze [27])**.** *Consider a group presentation $\langle S|R\rangle \cong G$. Then:*

- *Let r be a word in S which is zero in $\langle S|R\rangle$. Then $\langle S|R \cup \{r\}\rangle \cong G$.*
- *Let w be a word in S, and let g be fresh. Then $\langle S \cup \{g\}|R \cup \{g^{-1}w\}\rangle \cong G$.*

We refer to the passage from one presentation to another in either of these ways, in either direction, as a Tietze transformation.

We now establish several transformations of 3-presentations:

Lemma 2.3.8. *Let $\langle S|R\rangle \cong \mathbb{Z}^n$ be a 3-presentation, and suppose $g = 0$ in $\langle S|R\rangle$, where $g \in S$. Then we obtain a 3-presentation $\langle S'|R'\rangle \cong \mathbb{Z}^n$ where:*

- $S' = S \setminus \{g\}$.
- *R' is obtained from R by removing g wherever it appears in relations $r \in R$. (For example, $ghi \in R$ becomes $hi \in R'$.)*

Proof. We apply Tietze transformations:

- Add the redundant relation g to R to obtain R'.
- Remove g wherever it appears in relations $r \in R'$, except in the relation $g \in R'$. This is valid since $g = 0$ in $\langle S|R' \rangle$ by the relation $g \in R'$. (Each such removal is two Tietze transformations, adding and removing a relation.)
- Remove the generator g, along with the relation g.

This gives the desired 3-presentation of $\mathbb{Z}^n$. □

Lemma 2.3.9. *Let $\langle S|R \rangle \cong \mathbb{Z}^n$ be a 3-presentation, and suppose $g^a h^b = 0$ in $\langle S|R \rangle$, where $g, h \in S$ are distinct, $a, b \neq 0$, and a, b are relatively prime. Then we obtain a 3-presentation $\langle S'|R' \rangle \cong \mathbb{Z}^n$ where:*

- *$S' = S \cup \{i\} \setminus \{g, h\}$, where i is a fresh generator.*
- *R' is obtained from R by replacing g with i^b and h with i^{-a} wherever they appear in relations $r \in R$.*

Proof. There exist $c, d \in \mathbb{Z}$ with $ac + bd = 1$. We apply Tietze transformations:

- Add the relation $g^a h^b$, which is redundant by assumption.
- Add a generator i, along with the relation $i^{-1} g^d h^{-c}$, to obtain a new 3-presentation $\phi' \colon \langle S'|R' \rangle \cong \mathbb{Z}^n$.
- Add the relation $g^{-1} i^b$, which is redundant since
$$i^b = g^{bd} h^{-bc} = g^{1-ac} h^{-bc} = g(g^a h^b)^{-c} = g$$
in $\langle S'|R' \rangle$. (We use commutativity of g, h in $\langle S'|R' \rangle$, which follows from commutativity of $\phi'(g), \phi'(g)$ in $\mathbb{Z}^n$.)
- Similarly, add the relation $h^{-1} i^{-a}$, which is redundant since
$$i^{-a} = g^{-ad} h^{ac} = g^{-ad} h^{1-bd} = h(g^a h^b)^{-d} = h$$
in $\langle S'|R' \rangle$.
- Replace g with i^b and h with i^{-a} wherever they appear in relations $r \in R'$ (i.e. in all relations other than the new relations $g^{-1} i^b, h^{-1} i^{-a}$).
- Remove the generators g, h, along with the relations $g^{-1} i^b, h^{-1} i^{-a}$.
- Remove the relation $i^{-1} g^d h^{-c}$, which is now $i^{-1} i^{bd} i^{ac} = 0$.
- Remove the relation $g^a h^b$, which is now $(i^b)^a (i^{-a})^b = 0$.

This gives the desired 3-presentation of $\mathbb{Z}^n$. □

Lemma 2.3.10. *Let $\phi \colon \langle S|R \rangle \cong \mathbb{Z}^n$ be a 3-presentation with $|S|$ minimal. Then for each nonempty $r \in R$, we have $r \rightsquigarrow g^a h^b i^c$, where $g, h, i \in S$ are distinct and $a, b, c \neq 0$, and* $\dim r = 2$.

Proof. Let $r \rightsquigarrow \prod_{i=1}^{k} g_i^{a_i}$ by Remark 2.3.2. By Lemma 2.3.8, no g_i are zero in $\langle S|R\rangle$, which implies $k \neq 1$. By Lemma 2.3.9, no distinct g_i, g_j have $\phi(g_i)$, $\phi(g_j)$ in a common one-dimensional subspace of $\mathbb{R}^n$, which implies $k \neq 2$.

Hence $k = 3$, so $r \rightsquigarrow g^a h^b i^c$ for $g, h, i \in S$ distinct and $a, b, c \neq 0$. Then the considerations above imply $\dim\{g, h\} = 2$, so $\dim r \geq 2$. Since $\phi(g), \phi(h), \phi(i)$ are dependent in $\mathbb{R}^n$, we have $\dim r = 2$. □

Our next transformation requires the notion of a *sparse* set of relations:

Definition 2.3.11. Let $\langle S|R\rangle \cong \mathbb{Z}^n$ be a 3-presentation, and let $S' \subseteq S$, $R' \subseteq R$. Then define the set $R'[S'] \subseteq R'$ as

$$R'[S'] = \{r \in R' : r \rightsquigarrow w, \text{ and } w \text{ uses only generators in } S'\}.$$

Definition 2.3.12. Let $\phi\colon \langle S|R\rangle \cong \mathbb{Z}^n$ be a 3-presentation, and let $R' \subseteq R$.

- For $S' \subseteq S$ with $\dim S' = 2$, R' is *sparse on* S' if $|R'[S']| \leq |S'| - 1$.
- R' is *sparse* if R' is sparse on all $S' \subseteq S$ with $\dim S' = 2$.
- A set $S' \subseteq S$ is *critical for* R' if $\dim S' = 2$ and $|R'[S']| = |S'| - 1$.

Remark 2.3.13. *Let $\phi\colon \langle S|R\rangle \cong \mathbb{Z}^n$ be a 3-presentation with $|S|$ minimal, so that Lemma 2.3.10 applies. Suppose $R' \subseteq R$ is sparse, and $S', S'' \subseteq S$ are critical for R'. If $R[S'] \cap R[S''] \neq \emptyset$, then $S' \cup S''$ is also critical for R'.*

Proof. Let $r \in R[S'] \cap R[S'']$, and write $r \rightsquigarrow g^a h^b i^c$ by Lemma 2.3.10. Then the set $\{\phi(g), \phi(h), \phi(i)\}$ spans a 2-dimensional subspace $U \subseteq \mathbb{R}^n$. Since $\dim S' = \dim S'' = 2$, we have $\operatorname{span}(\phi(S')) = \operatorname{span}(\phi(S'')) = U$. Then

$$U \subseteq \operatorname{span}(\phi(S' \cap S'')) \subseteq \operatorname{span}(\phi(S')) = U,$$

so $\operatorname{span}(\phi(S' \cap S'')) = U$, and also $\operatorname{span}(\phi(S' \cup S'')) = U + U = U$. In particular, $\dim(S' \cap S'') = \dim(S' \cup S'') = 2$. Therefore, we have

$$\begin{aligned}
|R'[S' \cup S'']| &= |R'[S']| + |R'[S'']| - |R'[S' \cap S'']| \\
&\geq (|S'| - 1) + (|S''| - 1) - (|S' \cap S''| - 1) \\
&\geq |S' \cup S''| - 1.
\end{aligned}$$

Hence $S' \cup S''$ is critical for R'. □

Corollary 2.3.14. *Let $\phi\colon \langle S|R\rangle \cong \mathbb{Z}^n$ be a 3-presentation with $|S|$ minimal, so that Lemma 2.3.10 applies. Suppose $R' \subseteq R$ is sparse. Then there exists a collection $\mathcal{C}$ of certain critical sets $S' \subseteq S$ for R', such that:*

(1) If $S'' \subseteq S$ is critical for R', then there exists $S' \in \mathcal{C}$ with $S'' \subseteq S'$.

(2) If $S', S'' \in \mathcal{C}$, then $R[S'] \cap R[S''] = \emptyset$.

Proof. First take $\mathcal{C} = \{S' \subseteq S : S' \text{ critical for } R'\}$; then (1) holds. If $S', S'' \in \mathcal{C}$ with $R[S'] \cap R[S''] \neq \emptyset$, then by Remark 2.3.13, $S' \cup S''$ is critical for R'. Then consider removing S', S'' from $\mathcal{C}$, and adding $S' \cup S''$ if it is not present.

While (2) fails, apply the step above repeatedly. Each step preserves (1) and reduces $|\mathcal{C}|$, so this process terminates with $\mathcal{C}$ such that (1), (2) both hold. □

Lemma 2.3.15. *Let $\phi\colon \langle S|R\rangle \cong \mathbb{Z}^n$ be a 3-presentation with $|S|$ minimal, so that Lemma 2.3.10 applies. Partition R as $R = R_s \sqcup R_e \sqcup R_o$ (mnemonic: "sparse," "extra," "other"), such that R_s is sparse, and for each $r \in R_e$ with $r \rightsquigarrow g^a h^b i^c$, we have $\{g, h, i\} \subseteq S'$ for some critical $S' \subseteq S$ for R_s. Then we obtain a 3-presentation $\langle S'|R'\rangle \cong \mathbb{Z}^n$ where:*

- *S' includes all generators in S.*
- *R' includes all relations in R_o.*
- *$|R'| - |S'| = |R_s| + |R_o| - |S|$.*

Proof. Obtain a collection $\mathcal{C}$ of critical sets for R_s via Corollary 2.3.14. Then for each $S'' \in \mathcal{C}$, consider the integer span of $\phi(S'')$ in $\mathbb{Z}^n$, that is, the set

$$\Lambda = \left\{ \sum_{i=1}^{k} a_i \phi(g_i) : k \in \mathbb{N},\, a_i \in \mathbb{Z},\, g_i \in S'' \right\}.$$

We have $\Lambda \cong \mathbb{Z}^2$ (see [9], Theorem 1.12.3), so Λ has a basis $\{x_1, x_2\}$. We apply Tietze transformations (for each $S'' \in \mathcal{C}$) to $\langle S|R\rangle$. (We will introduce some relations with more than three generators, but we remove these later.)

- For each $j \in [2]$, write $x_j = \sum_i a_i \phi(g_i)$ for $a_i \in \mathbb{Z}$, $g_i \in S''$. Then add a generator h_j, along with the relation $h_j^{-1} \prod_i g_i^{a_i}$, to obtain $\phi'\colon \langle S'|R'\rangle \cong \mathbb{Z}^n$. Note that
$$\phi'(h_j) = \phi\left(\prod_i g_i^{a_i}\right) = \sum_i a_i \phi(g_i) = x_j.$$
- For each $g \in S''$, write $\phi(g) = \sum_j b_j x_j$ for $b_j \in \mathbb{Z}$. Then add the relation $g^{-1} \prod_j h_j^{b_j}$, which is redundant since
$$\phi'\left(g^{-1} \prod_j h_j^{b_j}\right) = -\phi(g) + \sum_j b_j \phi'(h_j) = 0,$$
where we use $\phi'(h_j) = x_j$ in the last step.
- Add a generator h_*, along with the relation $h_*^{-1} h_1 h_2$.
- Add the relation $h_*^{-1} h_2 h_1$, which is redundant since $\mathbb{Z}^n$ (and hence our current $\langle S'|R'\rangle \cong \mathbb{Z}^n$) is abelian. Note that the relation $h_1 h_2 h_1^{-1} h_2^{-1}$ is now implied by the relations $h_*^{-1} h_1 h_2, h_*^{-1} h_2 h_1$.

- Remove all relations $r \in R[S'']$, which are now redundant. To see this, first rewrite r in terms of only the h_j, via the relations $g^{-1} \prod_j h_j^{b_j}$. Then rewrite r as $\prod_j h_j^{b_j}$ for $b_j \in \mathbb{Z}$, via the relations $h_i h_j h_i^{-1} h_j^{-1}$. Applying ϕ', we obtain $\sum_j b_j x_j = 0$, so $b_j = 0$ by the lattice structure of $\Lambda \cong \mathbb{Z}^2$. Hence we have rewritten r as the empty word, so r is redundant.
- Remove the relations $h_j^{-1} \prod_i g_i^{b_i}$ added in the first step, which are now redundant, since we may rewrite any such relation in terms of only the h_j, and then apply the previous argument.

After applying these steps for each $S'' \in \mathcal{C}$, we call the resulting 3-presentation $\langle S' | R' \rangle$. For each $S'' \in \mathcal{C}$, we have added three generators and a net of $|S''| - |R[S'']| + 2$ relations. By definition of $\mathcal{C}$, the sets $R[S'']$ are disjoint for distinct $S'' \in \mathcal{C}$. Hence we have

$$\begin{aligned} |R'| - |S'| &= |R| - |S| + \sum_{S'' \in \mathcal{C}} (|S''| - |R[S'']| - 1) \\ &= |R| - |S| + \sum_{S'' \in \mathcal{C}} (|S''| - |R_s[S'']| - 1) - \sum_{S'' \in \mathcal{C}} |R_e[S'']| \\ &= |R| - |S| - |R_e| \\ &= |R_s| + |R_o| - |S|. \end{aligned}$$

This completes the proof. □

We need one more transformation of presentations:

Lemma 2.3.16. *Let $\phi \colon \langle S | R \rangle \cong \mathbb{Z}^n$ be a 3-presentation, let $S' \subseteq S$, and let $d = \dim S'$. Then we obtain a presentation $\langle S'' | R'' \rangle \cong \mathbb{Z}^{n-d}$ where:*

- $S'' = S \setminus S'$.
- *R'' is obtained from R by adding d relations to form R', then removing each $g \in S'$ wherever it appears in relations $r \in R'$.*

Proof. Let $U = \operatorname{span}(\phi(S'))$ in $\mathbb{R}^n$. Then $U \cap \mathbb{Z}^n$ is a lattice of dimension d, so we may take a basis $\{x_1, \ldots, x_d\}$ of $U \cap \mathbb{Z}^n$, and extend to a basis $\{x_1, \ldots, x_n\}$ of $\mathbb{Z}^n$ (see Chapter 2, Lemma 4 of [24]). Then for each $i \in [d]$, let w_i be a word in $\langle S \rangle$ with $\phi(w_i) = x_i$ in $\mathbb{Z}^n$. Let $R' = R \cup \{w_1, \ldots, w_d\}$.

We claim $\langle S | R' \rangle \cong \mathbb{Z}^{n-d}$. To prove this, we will construct an isomorphism $\psi \colon \langle S | R' \rangle \to \mathbb{Z}^{n-d}$. Let $p \colon \mathbb{Z}^n \to \mathbb{Z}^{n-d}$ be the projection to the last $n - d$ coordinates under the basis $\{x_1, \ldots, x_n\}$; more precisely,

$$p\left(\sum_{i=1}^{n} a_i x_i\right) = \sum_{i=d+1}^{n} a_i y_i,$$

where $\{y_{d+1}, \ldots, y_n\}$ is a basis for $\mathbb{Z}^{n-d}$. Note that p is linear. Now define ψ on S by $\psi(g) = p(\phi(g))$ for all $g \in S$, and extend ψ to $\langle S \rangle$ by the universal property of the free group. Then for any word $w = \prod_i g_i^{a_i}$ in $\langle S \rangle$, we have

$$\psi(w) = \sum_i a_i \psi(g_i) = \sum_i a_i p(\phi(g_i)) = p\left(\sum_i a_i \phi(g_i)\right) = p(\phi(w)).$$

In particular, for $r \in R$, we have $\psi(r) = p(\phi(r)) = p(0) = 0$. For the w_i above, we have $\psi(w_i) = p(\phi(w_i)) = p(x_i) = 0$. Therefore, ψ is well-defined on $\langle S|R' \rangle$.

To show ψ is injective, suppose $\psi(w) = 0$ for $w \in \langle S \rangle$. Then $p(\phi(w)) = 0$, so $\phi(w) = \sum_{i=1}^d a_i x_i$ for some $a_i \in \mathbb{Z}$. Then $\phi(w) = \phi(\prod_{i=1}^d w_i^{a_i})$ so $w = \prod_{i=1}^d w_i^{a_i}$ in $\langle S|R \rangle$ by the injectivity of ϕ. Since $R \subseteq R'$, we have $w = \prod_{i=1}^d w_i^{a_i}$ in $\langle S|R' \rangle$ also. But since $w_i \in R$, this implies $w = 0$ in $\langle S|R' \rangle$.

To show ψ is surjective, it suffices to show that for each $d < i \leq n$, there exists $w \in \langle S \rangle$ with $\psi(w) = y_i$. By the surjectivity of ϕ, take w with $\phi(w) = x_i$. Then $\psi(w) = p(\phi(w)) = p(x_i) = y_i$ as desired. Hence $\langle S|R' \rangle \cong \mathbb{Z}^{n-d}$.

Now all generators $g \in S'$ have $g = 0$ in $\langle S|R' \rangle$, so repeated application of Lemma 2.3.8 gives the desired result. □

Next, we need the following Sylvester-Gallai-type result. This is only a slight modification of the results in [6, 11, 10]; we translate the average case result in [11] from an affine setting to a linear one (as in [10]), with a guarantee on $|E'|$:

Theorem 2.3.17. *Let $V \subseteq \mathbb{R}^d$ be a finite set of points, and let E be a finite set of (not necessarily distinct) triples $\{u, v, w\}$ of distinct points $u, v, w \in V$ lying in a common 2-dimensional subspace of $\mathbb{R}^d$, so that (V, E) forms a 3-uniform hypergraph. Suppose that for each induced subhypergraph (V', E') of (V, E) with* $\dim(\text{span } V') \leq 2$, *we have* $|E'| \leq |V'| - 1$. *Then for* $\lambda > 0$, *there exists an induced subhypergraph* (V', E') *of* (V, E) *with* $|E| - |E'| < \lambda|V|$, *and*

$$\dim(\text{span } V') \leq 12|V|/\lambda.$$

Proof. Following the proof of Theorem 13 in [6], consider (V, E) as a 3-uniform hypergraph, and repeatedly remove vertices of degree less than λ. This removes less than $\lambda|V|$ edges, so we obtain a sub-hypergraph (V', E') with $|E| - |E'| < \lambda|V|$ and minimum degree at least λ.

Fix $u \in V'$; the neighborhood $N(u)$ in (V', E') forms a graph $G(u)$, where we consider two vertices $v, w \in N(u)$ adjacent iff $\{u, v, w\} \in E'$. If $v, w \in N(u)$ are adjacent in $G(u)$, then w lies in span$(\{u, v\}) \subseteq \mathbb{R}^d$. Therefore, if $\{v_1, \ldots, v_k\}$ form a component C of $G(u)$, then $U = \{u, v_1, \ldots, v_k\}$ has dim(span $U) \leq 2$, so the number of triples in E' using only points in U is at most k. Hence the number of edges in C is at most k. Summing over components C, the number of neighbors of v in (V', E') is at least $\deg_{(V', E')} v \geq \lambda$.

Now choose a nonzero vector $\vec{n} \in \mathbb{R}^d$ not orthogonal to any $v \in V'$, and define an affine hyperplane $H = \{\vec{x} \in \mathbb{R}^d : \vec{x} \cdot \vec{n} = 1\}$. Then to each $v \in V'$ we associate the unique point $\tilde{v} \in \text{span}(\{v\}) \cap H$; note that u, v, w lie in a common two-dimensional subspace of $\mathbb{R}^d$ iff $\tilde{u}, \tilde{v}, \tilde{w}$ lie on a common line in H.

Then the set $\tilde{V'} = \{\tilde{v} : v \in V'\}$ meets the criteria for a δ-SG configuration in [11] with $\delta = \lambda/|V|$, except that $\tilde{v}, \tilde{w}$ are not necessarily distinct for distinct $v, w \in V'$. But the proof of Theorem 5.1 in [11] still holds if points in a δ-SG configuration are allowed to repeat, since the design matrix construction in Lemma 5.2 in [11] does not rely on points being distinct. Therefore, the affine dimension of $\tilde{V'}$ is at most $12/\delta$, so $\dim(\text{span } V') \leq 12|V|/\lambda$ as desired. □

The proof above holds even if points in V are not required to be distinct, so we obtain the following more general result:

Theorem 2.3.18. *Let V be a finite set with a function $\phi\colon V \to \mathbb{R}^d$, and let E be a finite set of (not necessarily distinct) triples $\{u, v, w\}$ of distinct points $u, v, w \in V$ with $\phi(u), \phi(v), \phi(w)$ lying in a common 2-dimensional subspace of $\mathbb{R}^d$, so that (V, E) forms a 3-uniform hypergraph. Suppose that for each induced subhypergraph (V', E') of (V, E) with $\dim(\text{span } \phi(V')) \leq 2$, we have $|E'| \leq |V'| - 1$. Then for $\lambda > 0$, there exists an induced subhypergraph (V', E') of (V, E) with $|E| - |E'| < \lambda|V|$, and*

$$\dim(\text{span } \phi(V')) \leq 12|V|/\lambda.$$

Now we prove our bound on the size of 3-presentations of $\mathbb{Z}^n$:

Theorem 2.3.19. *If $\langle S|R\rangle \cong \mathbb{Z}^n$ is a 3-presentation, then $|S| = \Omega(n^{3/2})$.*

Proof. Fix an isomorphism $\phi\colon \langle S|R\rangle \to \mathbb{Z}^n$. Assume that $|S|$ is minimal; then by Lemma 2.3.10, for each $r \in R$ we have $r \rightsquigarrow g^a h^b i^c$ for $g, h, i \in S$ distinct, and $\dim r = 2$. Then let R' be an inclusion-wise maximal sparse subset of R.

Now let $k = |S|$, let $c > 0$ be a constant to be determined later, and apply Theorem 2.3.18 with $V = S$, $E = \{\{g, h, i\} : r \rightsquigarrow g^a h^b i^c,\ r \in R'\}$, and $\lambda = ck/n$, to obtain $S' \subseteq S$ such that:

(1) $|R' \setminus R[S']| \leq ck^2/n$.

(2) $\dim S' \leq 12|V|/\lambda = 12n/c$.

If there exists $g \in S \setminus S'$ with $\dim(S' \cup \{g\}) = \dim S'$, then we may replace S' with $S' \cup \{g\}$, preserving (1) and (2). Therefore, we may assume that for each $g \in S \setminus S'$, we have $\phi(g) \notin \text{span } \phi(S')$.

Now partition R as $R = R_s \sqcup R_e \sqcup R_o$, where:

- $R_s = R' \setminus R[S']$.
- $R_e = (R \setminus R') \setminus R[S']$.
- $R_o = R[S']$.

Note that $|R_s| \leq ck^2/n$ by the above. R_s is sparse, since sparseness is closed under taking subsets. Also, for each $r \in R_e$ with $r \rightsquigarrow g^a h^b i^c$, we have $\{g, h, i\} \subseteq S''$ for some critical $S'' \subseteq S$ for R', since $r \notin R'$ and R' is maximal. But since $r \notin R[S']$, we have $\{g, h, i\} \not\subseteq S'$; assume WLOG $g \notin S'$. Then $\phi(g) \notin \text{span } \phi(S')$

by the above, so span $\phi(S'') \not\subseteq$ span $\phi(S')$. Then $R[S''] \cap R[S'] = \emptyset$, since any $r' \in R[S'']$ determines the 2-dimensional subspace span $\phi(S'')$. Therefore, S'' is also critical for R_s. Hence we may apply Lemma 2.3.15 to $\langle S|R\rangle$, to obtain a 3-presentation $\langle S''|R''\rangle \cong \mathbb{Z}^n$ with $|R''| - |S''| = |R_s| + |R_o| - |S|$.

Finally, let $d = \dim S'$, and apply Lemma 2.3.16 to $\langle S''|R''\rangle$ using $S' \subseteq S''$, to obtain $\langle S'''|R'''\rangle \cong \mathbb{Z}^{n-d}$. Then remove all relations in R''' arising from relations in $R_o = R[S']$, which are now trivial, to obtain $\langle S'''|R''''\rangle \cong \mathbb{Z}^{n-d}$. Then

$$\begin{aligned}
|R''''| - |S'''| &= (|R'''| - |R_o|) - |S'''| \\
&= (|R''| + d - |R_o|) - (|S''| - |S'|) \\
&= (|R''| - |S''| - |R_o|) + d + |S'| \\
&= (|R_s| - |S|) + d + |S'| \\
&= |R_s| + d - |S \setminus S'| \\
&\leq ck^2/n + d.
\end{aligned}$$

But by the bound def $\mathbb{Z}^m \leq m - \binom{m}{2}$, we have $|R''''| - |S'''| = \Omega((n-d)^2)$. Take $c = 24$; then $d \leq 12n/c = n/2$, so $n - d \geq n/2$. Hence $|R''''| - |S'''| = \Omega(n^2)$. Since $d \leq n/2$, we have $ck^2/n = \Omega(n^2)$. Therefore, $k = \Omega(n^{3/2})$ as desired. $\square$

Theorem 2.3.20. *A simplicial complex X with fundamental group $\pi_1(X) \cong \mathbb{Z}^n$ has at least $\Omega(n^{3/4})$ vertices.*

Proof. Let $f(n)$ be the minimum number of vertices in a simplicial complex X_n with fundamental group $\pi_1(X_n) \cong \mathbb{Z}^n$. By Lemma 2.3.3, for each n we obtain a 3-presentation $\langle S_n|R_n\rangle \cong \mathbb{Z}^n$ with $|S_n| \leq \binom{f(n)}{2}$. But $|S_n| = \Omega(n^{3/2})$, so $\binom{f(n)}{2} = \Omega(n^{3/2})$, hence $f(n) = \Omega(n^{3/4})$. $\square$

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References

[1] Karim Adiprasito, Sergey Avvakumov, and Roman Karasev. *A subexponential size $\mathbb{RP}^n$*. 2020. arXiv: `2009.02703 [math.CO]`.

[2] Sylwia Antoniuk, Ehud Friedgut, and Tomasz Łuczak. "A sharp threshold for collapse of the random triangular group". In: *Groups Geom. Dyn.* 11.3 (2017), pp. 879–890.

[3] Sylwia Antoniuk, Tomasz Łuczak, and Jacek Świątkowski. "Collapse of random triangular groups: a closer look". In: *Bull. Lond. Math. Soc.* 46.4 (2014), pp. 761–764.

[4] Sylwia Antoniuk, Tomasz Łuczak, and Jacek Światkowski. "Random triangular groups at density 1/3". In: *Compos. Math.* 151.1 (2015), pp. 167–178.

[5] Pierre Arnoux and Alexis Marin. "The Kühnel triangulation of the complex projective plane from the view point of complex crystallography. II". In: *Mem. Fac. Sci. Kyushu Univ. Ser. A* 45.2 (1991), pp. 167–244.

[6] Boaz Barak et al. "Fractional Sylvester-Gallai theorems". In: *Proc. Natl. Acad. Sci. USA* 110.48 (2013), pp. 19213–19219.

[7] Louis J. Billera and Anders Björner. "Face numbers of polytopes and complexes". In: *Handbook of discrete and computational geometry*. CRC Press Ser. Discrete Math. Appl. CRC, Boca Raton, FL, 1997, pp. 291–310.

[8] U. Brehm and W. Kühnel. "Combinatorial manifolds with few vertices". In: *Topology* 26.4 (1987), pp. 465–473.

[9] J. J. Duistermaat and J. A. C. Kolk. *Lie groups*. Universitext. Springer-Verlag, Berlin, 2000, pp. viii+344.

[10] Zeev Dvir and Guangda Hu. "Sylvester-Gallai for arrangements of subspaces". In: *Discrete Comput. Geom.* 56.4 (2016), pp. 940–965.

[11] Zeev Dvir, Shubhangi Saraf, and Avi Wigderson. "Improved rank bounds for design matrices and a new proof of Kelly's theorem". In: *Forum Math. Sigma* 2 (2014), Paper No. e4, 24.

[12] D. B. A. Epstein. "Finite presentations of groups and 3-manifolds". In: *Quart. J. Math. Oxford Ser. (2)* 12 (1961), pp. 205–212.

[13] Allen Hatcher. *Algebraic topology*. Cambridge University Press, Cambridge, 2002, pp. xii+544.

[14] J. D. Horton. "Room designs and one-factorizations". In: *Aequationes Math.* 22.1 (1981), pp. 56–63.

[15] Gil Kalai. "Enumeration of $\mathbb{Q}$-acyclic simplicial complexes". In: *Israel J. Math.* 45.4 (1983), pp. 337–351.

[16] W. Kühnel and G. Lassmann. "Combinatorial d-tori with a large symmetry group". In: *Discrete Comput. Geom.* 3.2 (1988), pp. 169–176.

[17] Wolfgang Kühnel. "Higher-dimensional analogues of Czászár's torus". In: *Results Math.* 9.1-2 (1986), pp. 95–106.

[18] Frank H. Lutz. *Triangulated Manifolds with Few Vertices: Combinatorial Manifolds*. 2005. eprint: `math/0506372`.

[19] Roger C. Lyndon and Paul E. Schupp. *Combinatorial group theory*. Classics in Mathematics. Reprint of the 1977 edition. Springer-Verlag, Berlin, 2001, pp. xiv+339.

[20] Eric Mendelsohn and Alexander Rosa. "One-factorizations of the complete graph—a survey". In: *J. Graph Theory* 9.1 (1985), pp. 43–65.

[21] R. C. Mullin and W. D. Wallis. "The existence of Room squares". In: *Aequationes Math.* 13.1-2 (1975), pp. 1–7.

[22] Satoshi Murai and Isabella Novik. "Face numbers and the fundamental group". In: *Israel J. Math.* 222.1 (2017), pp. 297–315.

[23] Andrew Newman. "Small simplicial complexes with prescribed torsion in homology". In: *Discrete Comput. Geom.* 62.2 (2019), pp. 433–460.

[24] Phong Q. Nguyen and Brigitte Vallée, eds. *The LLL algorithm*. Information Security and Cryptography. Survey and applications. Springer-Verlag, Berlin, 2010, pp. xiv+496.

[25] Petar Pavešić. "Triangulations with few vertices of manifolds with non-free fundamental group". In: *Proc. Roy. Soc. Edinburgh Sect. A* 149.6 (2019), pp. 1453–1463.

[26] Richard G. Swan. "Minimal resolutions for finite groups". In: *Topology* 4 (1965), pp. 193–208.

[27] Heinrich Tietze. "Über die topologischen Invarianten mehrdimensionaler Mannigfaltigkeiten". In: *Monatsh. Math. Phys.* 19.1 (1908), pp. 1–118.

Chapter 3

Simplicial complexes, finite projective planes, and colored configurations

To be submitted for publication.

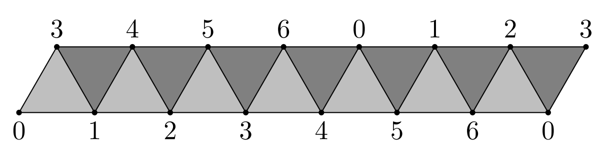
Abstract

In the 7-vertex triangulation of the torus, the 14 triangles can be partitioned as $T_1 \sqcup T_2$, such that each T_i represents the lines of a copy of the Fano plane $PG(2, \mathbb{F}_2)$. We generalize this observation by constructing, for each prime power q, a simplicial complex K with $q^2 + q + 1$ vertices and $2(q^2 + q + 1)$ facets consisting of two copies of $PG(2, \mathbb{F}_q)$.

Our construction works for any *colored k-configuration*, defined as a k-configuration whose associated bipartite graph G is connected and has a k-edge coloring $\chi\colon E(G) \to [k]$, such that for all $v \in V(G)$, $a, b, c \in [k]$, following edges of colors a, b, c, a, b, c from v brings us back to v. We give constructions of colored k-configurations from planar difference sets and commutative semifields. Then we give one-to-one correspondences between (1) Sidon sets of order 2 and size $k + 1$ in groups with order n, (2) linear codes with radius 1 and index n in A_k, and (3) colored $(k + 1)$-configurations with n points and n lines.

3.1 Introduction

The torus T^2 has a 7-vertex triangulation, arising from the following diagram:



To see this, identify any two vertices with the same label, and identify any two edges whose ends have the same label. Combinatorially, this identification produces a simplicial complex on 7 vertices. Topologically, this identification is equivalent to first identifying the leftmost and rightmost edges to obtain a cylinder, and then identifying the top and bottom circles to obtain T^2.

This triangulation K of T^2 has several notable properties:

- K has exactly 7 vertices. (In fact, K is vertex-minimal; any triangulation of T^2 has at least 7 vertices; see [11, 16].)
- K contains the Fano plane; the triangles pointing up,

  $$\{0,1,3\},\{1,2,4\},\{2,3,5\},\{3,4,6\},\{4,5,0\},\{5,6,1\},\{6,0,2\},$$

  can be viewed as the lines of the Fano plane on points $\{0,\ldots,6\}$. (Similarly for the triangles pointing down.)
- K is cyclic; the cyclic group $\mathbb{Z}_7$ acts on K by cyclically permuting labels.
- K is 2-neighborly; that is, each pair of vertices in K form an edge.

The Fano plane is also known as the projective geometry $PG(2,\mathbb{F}_2)$, where $\mathbb{F}_2$ is the field of order 2, and similar projective geometries $PG(2,\mathbb{F}_q)$ exist for any prime power q. Thus we ask:

Question 3.1.1. *Does the 7-vertex triangulation of T^2, along with its notable properties, generalize to prime power dimensions $q > 2$?*

Our main result is a construction (Theorem 3.5.2), which takes as its input a colored $(k+1)$-configuration $\mathcal{C}$ (see Definition 3.2.7), and produces a simplicial complex $K(\mathcal{C})$ with $\pi_1(K(\mathcal{C})) \cong \mathbb{Z}^k$. The complex $K(\mathcal{C})$ is not T^k in general, but can be made homeomorphic to T^k by adding vertices and faces.

For example, if $\mathcal{C}$ is $PG(2,\mathbb{F}_q)$, then we obtain the following:

Corollary 3.6.6. *Let q be a prime power. Then there exists a connected q-dimensional simplicial complex K with $\pi_1(K) \cong \mathbb{Z}^q$, such that:*

- *K has exactly q^2+q+1 vertices.*
- *K contains two copies of $PG(2,\mathbb{F}_q)$, each consisting of q^2+q+1 facets of K. These two copies fully describe K, in that these $2(q^2+q+1)$ facets are all of the facets of K, and every face of K is contained in a facet.*
- *K is cyclic; the cyclic group $\mathbb{Z}_{q^2+q+1}$ acts freely on K.*
- *K is 2-neighborly.*

In the general construction, the complex $K(\mathcal{C})$ contains one copy of $\mathcal{C}$, and one copy of its dual $\mathcal{C}^*$, which is obtained from $\mathcal{C}$ by switching points and lines. Since $PG(2,\mathbb{F}_q)$ is isomorphic to its dual, in this case we obtain two copies of $PG(2,\mathbb{F}_q)$. See Section 3.6 for details.

We make no claim of vertex-minimality. However, we note that the smallest known triangulations of T^k use $2^{k+1} - 1$ vertices ([15]; see [16]). Our complex K from Corollary 3.6.6 uses fewer vertices but lacks the full structure of T^k. We conjecture that K is vertex-minimal in the following sense:

Conjecture 3.1.2. *Suppose K is a simplicial complex on n vertices, such that $\pi_1(K) \cong \mathbb{Z}^k$, and such that K admits a free $\mathbb{Z}_n$-action. Then $n \geq k^2 + k + 1$, with equality attainable only for prime powers k.*

This conjecture implies that all cyclic planar difference sets have prime power order, an open problem in design theory (see [1], Chapter VII). In this way, our work gives a possible topological approach to finding obstructions to the existence of planar difference sets and finite projective planes.

Our construction is closely related to a construction of linear codes from Sidon sets (see [14]). In our construction, we assign labels to a k-dimensional lattice and then quotient according to that labeling. The labeling of the lattice can be viewed as a linear code in the lattice A_k defined in Section 3.4. We show a relationship between three mathematical structures:

Theorem 3.8.4. *We have one-to-one correspondences (up to isomorphism) between any two of the following three mathematical structures:*

(1) Pairs (G, B), where G is an abelian group with $|G| = n$, and B is a Sidon set of order 2 in G with $|B| = k + 1$.

(2) Linear codes $\mathcal{L}$ with radius 1 in A_k, with $|A_k/\mathcal{L}| = n$.

(3) Colored $(k + 1)$-configurations $\mathcal{C}$ with n points and n lines.

The relationship between the first two structures above is known [14]; our contribution is to introduce the third. As a result, we obtain new constructions of Sidon sets and linear codes (for example, from Theorem 3.7.3). Also, we raise the possibility of topological obstructions to the existence of Sidon sets and linear codes, via the simplicial complex $K(\mathcal{C})$ given by Theorem 3.5.2.

3.2 Colored k-configurations

Following Grünbaum [7], we define a k-configuration as a certain kind of k-regular incidence structure (where $k \in \mathbb{N}$):

Definition 3.2.1. A *k-configuration* consists of finite sets P, L (whose elements are called "points" and "lines," respectively), and an incidence relation $R \subseteq P \times L$, satisfying the following conditions:

(1) There do not exist distinct $p_1, p_2 \in P$ and distinct $l_1, l_2 \in L$ with $(p_i, l_j) \in R$ for all $i, j \in \{1, 2\}$. (That is, no two points are on more than one common line; equivalently, no two lines contain more than one common point.)

(2) Each point is incident with exactly k lines.

(3) Each line is incident with exactly k points.

For a k-configuration $\mathcal{C}$, we will often write P, L, R as $P(\mathcal{C}), L(\mathcal{C}), R(\mathcal{C})$.

A simple counting argument shows the following:

Remark 3.2.2. *In a k-configuration, the numbers of points and lines are equal.*

A k-configuration $\mathcal{C}$ is equivalent to a k-regular bipartite graph $G(\mathcal{C})$ with no 4-cycle; the two parts of the graph correspond to the sets P, L, and the edge set of the graph corresponds to the relation R. The "no 4-cycle" requirement corresponds to condition (1), and the k-regularity corresponds to conditions (2) and (3). We will often require $G(\mathcal{C})$ to be connected:

Definition 3.2.3. A k-configuration $\mathcal{C}$ is *connected* if $G(\mathcal{C})$ is connected.

Remark 3.2.4. *Let $\mathcal{C}$ be a k-configuration with $k > 0$, and suppose there exists a path in $G(\mathcal{C})$ between any two points of $\mathcal{C}$. Then $\mathcal{C}$ is connected.*

Proof. Since $k > 0$, each line in $\mathcal{C}$ is incident with at least one point. □

We now consider k-coloring the incidences of a k-configuration $\mathcal{C}$; that is, we consider functions $\chi\colon R(\mathcal{C}) \to [k]$, such that if two incidences $(p_1, l_1), (p_2, l_2)$ have $p_1 = p_2$ or $l_1 = l_2$, then $\chi(p_1, l_1) \neq \chi(p_2, l_2)$. This corresponds to the usual notion of a k-edge coloring of $G(\mathcal{C})$:

Definition 3.2.5. For any graph G, a *k-edge coloring* of G is a function $\chi\colon E(G) \to [k]$ that assigns distinct colors to distinct edges sharing a vertex.

Note that if G is k-regular, then G always has a k-edge coloring, since line graphs of bipartite graphs are perfect [13]. (More concretely, we can obtain χ by repeated applications of Hall's theorem, for example.) For our construction, we will need the k-edge coloring χ to satisfy an additional property:

Definition 3.2.6. Let G be a k-regular bipartite graph, and let χ be a k-edge coloring of G. For each vertex $v \in V(G)$ and each color $c \in [k]$, let $\phi_c(v)$ be the unique vertex $w \in V(G)$ such that vw is an edge of G with color $\chi(vw) = c$.

We say that χ has the *6-cycle property*, if, for each vertex $v \in V(G)$, and for each triple (a, b, c) of distinct colors $a, b, c \in [k]$, we have

$$(\phi_c \circ \phi_b \circ \phi_a \circ \phi_c \circ \phi_b \circ \phi_a)(v) = v.$$

(This equation means that following the unique path from v along edges with colors a, b, c, a, b, c, in order, produces a 6-cycle in G.)

Definition 3.2.7. A *colored k-configuration* is a connected k-configuration $\mathcal{C}$, along with a k-edge coloring $\chi(\mathcal{C})$ of $G(\mathcal{C})$ satisfying the 6-cycle property.

Our construction takes a colored k-configuration as its input; the construction involves assigning labels to the lattice A_n, and the 6-cycle property ensures that we assign labels consistently.

Given the definitions above, it is natural to ask, which k-configurations can be colored? This question has an analogue in Sidon sets and is likely difficult to answer in general (see Section 3.8). Still, in Sections 3.6 and 3.7 we give several positive results in this direction. For example, $PG(2, \mathbb{F}_q)$, which is a $(q+1)$-configuration, can be colored (Corollary 3.6.5).

Duality

The duality on k-configurations extends to colored k-configurations:

Definition 3.2.8. Let $\mathcal{C}$ be a colored k-configuration. Then we obtain a *dual* colored k-configuration $\mathcal{C}^*$ as follows:

- The points of $\mathcal{C}^*$ are the lines of $\mathcal{C}$.
- The lines of $\mathcal{C}^*$ are the points of $\mathcal{C}$.
- A point p and line l are incident in $\mathcal{C}^*$ if p, l are incident in $\mathcal{C}$.
- The color of the incidence (p, l) in $\mathcal{C}^*$ is the color of (l, p) in $\mathcal{C}$.

Group actions

Our next goal is to define group actions on colored k-configurations:

Definition 3.2.9. A *homomorphism* $\psi \colon \mathcal{C} \to \mathcal{D}$ of colored k-configurations $\mathcal{C}, \mathcal{D}$ consists of functions $\psi_P \colon P(\mathcal{C}) \to P(\mathcal{D})$, $\psi_L \colon L(\mathcal{C}) \to L(\mathcal{D})$, $\psi_\chi \colon [k] \to [k]$, with the following properties:

- If $(p, l) \in R(\mathcal{C})$, then $(\psi_P(p), \psi_L(l)) \in R(\mathcal{D})$.
- If $(p, l) \in R(\mathcal{C})$, then $\chi(\mathcal{D})(\psi_P(p), \psi_L(l)) = \psi_\chi(\chi(\mathcal{C})(p, l))$.

In other words, ψ_P, ψ_L preserve incidences and color classes of incidences.

Then a group action ρ by a group G on a colored k-configuration $\mathcal{C}$ is, as usual, a group homomorphism $\rho \colon G \to \text{Aut}(\mathcal{C})$. Note that ρ consists of group actions ρ_P and ρ_L by G on $P(\mathcal{C}), L(\mathcal{C})$ preserving incidences and color classes of incidences.

3.3 Finite projective planes

In our definition of k-configuration, condition (1) says that any two points are on at most one common line, and vice versa. By replacing “at most one” with “exactly one,” we obtain a definition (though not the usual one) of a finite projective plane. We recall the usual definition [8] of a projective plane:

Definition 3.3.1. A *projective plane* consists of sets P, L (whose elements are called "points" and "lines," respectively), and an incidence relation $R \subseteq P \times L$, satisfying the following conditions:

(1) For distinct $p_1, p_2 \in P$, there exists a unique $l \in L$ with $(p_1, l), (p_2, l) \in R$.

(2) For distinct $l_1, l_2 \in L$, there exists a unique $p \in P$ with $(p, l_1), (p, l_2) \in R$.

(3) There exist distinct $p_1, p_2, p_3, p_4 \in P$, such that for each $l \in L$, at most two of the four pairs (p_1, l), (p_2, l), (p_3, l), (p_4, l) are in R.

We say that the projective plane is *finite* if P and L are finite.

In a finite projective plane, there exists an integer q, called the *order* of the finite projective plane, such that each point is incident with exactly $q + 1$ lines, and each line is incident with exactly $q + 1$ points (see [20]). It follows by a counting argument that the plane has exactly $q^2 + q + 1$ points and lines.

The typical example of a finite projective plane is $PG(2, \mathbb{F}_q)$:

Definition 3.3.2. Let q be a prime power. Then $PG(2, \mathbb{F}_q)$ is a finite projective plane of order q, defined with reference to the vector space $\mathbb{F}_q^3$ over $\mathbb{F}_q$:

- Let P be the set of one-dimensional subspaces of $\mathbb{F}_q^3$.
- Let L be the set of two-dimensional subspaces of $\mathbb{F}_q^3$.
- Let $R \subseteq P \times L$ be the set of pairs (p, l) of subspaces p, l with $p \subseteq l$.

(Conditions (1) and (2) follow from the identity $\dim(U + V) + \dim(U \cap V) = \dim U + \dim V$, and for condition (3) we may take the four one-dimensional subspaces spanned by $(1, 0, 0), (0, 1, 0), (0, 0, 1), (1, 1, 1)$.)

It is an open problem whether there exists a finite projective plane whose order q is not a prime power. The strongest negative result is given by Bruck & Ryser [4]; if the order q of a finite projective plane has $q \equiv 1, 2 \pmod 4$, then q is a sum of two squares.

To clarify the connection with k-configurations, note that any finite projective plane with order q is a $(q + 1)$-configuration. Conversely, any $(q + 1)$-configuration with $q \geq 2$ satisfying conditions (1) and (2) is a finite projective plane; this follows from Hall's characterization of degenerate planes [8].

3.4 The lattice A_n and the simplicial complex K_n

We first introduce the lattice A_n, which is well known from the sphere packing literature (see [5]):

Definition 3.4.1. For $n \geq 0$, the lattice A_n is defined by

$$A_n = \left\{ (x_1, \ldots, x_{n+1}) \in \mathbb{Z}^{n+1} : \sum_{i=1}^{n+1} x_i = 0 \right\}.$$

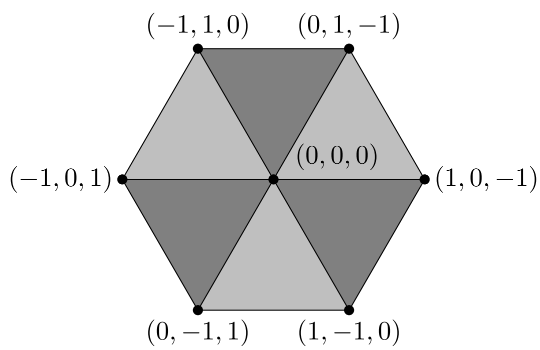
For example, A_2 is equivalent to the hexagonal lattice, and A_3 is equivalent to the face-centered cubic lattice. Locally, A_n has the structure of the expanded simplex (see [6]). We may consider A_n a metric space by using a scaled ℓ_1-distance, $d(\vec{x}, \vec{y}) = \|\vec{x} - \vec{y}\|_1/2$. (This metric d can be viewed as a graph metric, where we consider $\vec{x}, \vec{y}$ to be adjacent if $\|\vec{x} - \vec{y}\|_1 = 2$.)

We now introduce a simplicial complex structure K_n on A_n; the simplicial complex K_n is known as the *Rips complex* of diameter 1 of A_n [21].

Definition 3.4.2. For $n \geq 0$, the simplicial complex K_n is defined as follows:

- The vertices $V(K_n)$ of K_n are the points of A_n.
- The faces of K_n are the sets F with $d(\vec{x}, \vec{y}) = 1$ for all distinct $\vec{x}, \vec{y} \in F$.

Note that K_n is a flag simplicial complex; that is, if F is a set of vertices of K_n, and all pairs of vertices in F are edges of K_n, then F is a face of K_n. In this way, K_n is determined by its 1-skeleton. For K_2, the facets are exactly the triangles of the hexagonal lattice; here is $(0, 0, 0)$ and its neighbors:

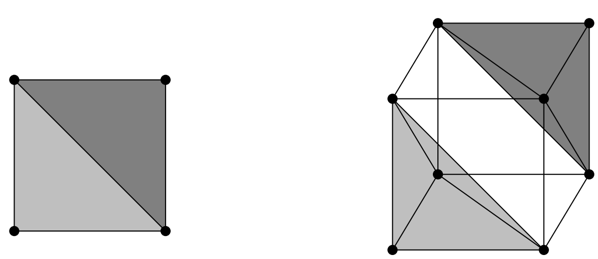


All six shaded triangles are facets of K_2; the light and dark shading indicate "positive" and "negative" facets, respectively, as defined below.

To gain intuition for A_n and K_n, consider the projection $p\colon \mathbb{R}^{n+1} \to \mathbb{R}^n$ onto the first n coordinates, that is,

$$p(x_1, \ldots, x_{n+1}) = (x_1, \ldots, x_n).$$

Then p carries A_n isomorphically (as an additive group) to $\mathbb{Z}^n$, and embeds K_n into $\mathbb{R}^n$. For example, we draw the facets of $p(K_2)$ inside the unit square of $\mathbb{R}^2$, and the facets of $p(K_3)$ inside the unit cube of $\mathbb{R}^3$:



We see that K_2 is a tiling of the plane $\{(x,y,z) \in \mathbb{R}^3 : x+y+z=0\}$, but for $n > 2$, K_n does not fill the hyperplane it spans. (This is why our construction will not generally give a trianguation of T^n.)

Remark 3.4.3. *The simplicial complex K_n has the following properties:*

- *K_n has dimension n.*
- *The facets of K_n are of the form*

$$\{\vec{x}+\vec{e}_i\}_{i\in[n+1]} \qquad \text{or} \qquad \{\vec{x}-\vec{e}_j\}_{j\in[n+1]}$$

  for fixed $\vec{x} \in \mathbb{Z}^{n+1}$ with sum of coordinates -1 (in the first case) or 1 (in the second). We call facets of the first type "positive," and facets of the second type "negative." We call $\vec{x}$ the "root" of the facet in either case. (Here the $\vec{e}_i$ are the standard basis vectors in $\mathbb{R}^{n+1}$.)
- *Each vertex of K_n is incident with exactly $n+1$ facets of each type.*
- *Each face F of K_n is contained in a facet of K_n. If $\dim F \geq 2$, then this facet is unique.*

Proof. Let F be a face of K_n, and assume $\vec{0} \in F$. Then each other vertex of F is of the form $\vec{e}_i - \vec{e}_j$ for distinct $i,j \in [n+1]$. Denote such a vertex by the ordered pair (i,j). If F contains two nonzero vertices $(i_1,j_1),(i_2,j_2)$, then $i_1 = i_2$ or $j_1 = j_2$. It follows that the nonzero vertices of F are of the form

$$\{(i_1,j),\ldots,(i_k,j)\} \qquad \text{or} \qquad \{(i,j_1),\ldots,(i,j_k)\}.$$

In the first case, we get a positive facet rooted at $-\vec{e}_j$; in the second case, we get a negative facet rooted at $\vec{e}_i$. The rest follows. □

Remark 3.4.4. *The group action of A_n on itself by addition induces a group action of A_n on the simplicial complex K_n.*

Proof. By the translational symmetry of K_n, addition by any $\vec{x} \in A_n$ takes any face of K_n to another face of K_n. □

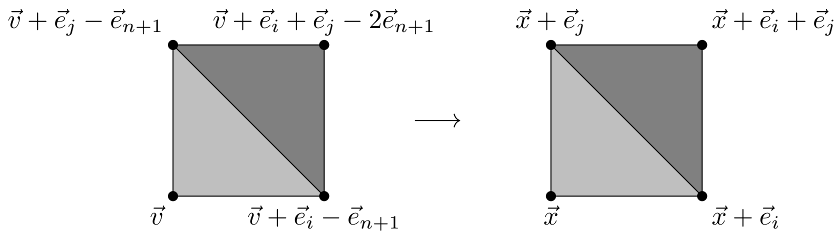
Recall that a simplicial complex K is *simply connected* if its geometric realization is connected and has trivial fundamental group. We saw above that the facets of K_3 fail to fill 3-dimensional space; however, $p(K_3)$ does contain the 2-skeleton of (a CW complex homeomorphic to) $\mathbb{R}^3$. Hence K_3 is simply connected. This generalizes to K_n for arbitrary n:

Remark 3.4.5. *The simplicial complex K_n is simply connected.*

Proof. The space $\mathbb{R}^n$ has a CW complex structure X; for each $\vec{x} \in \mathbb{Z}^n$, $I \subseteq [n]$, the CW complex X has a $|I|$-cell given by

$$f_{\vec{x},I} = \left\{ \vec{x} + \sum_{i \in I} \lambda_i \vec{e}_i : \lambda_i \in [0,1] \right\}.$$

We claim each 2-cell $f_{\vec{x},\{i,j\}}$ is present in $p(K_n)$ as the union of two triangles. To see this, let $\vec{v} \in A_n$ be the unique vector with $p(\vec{v}) = \vec{x}$. Then $f_{\vec{x},\{i,j\}}$ lies in the image $p(K_n)$ as follows:



Moreover, all 1-dimensional faces of K_n appear in this diagram for some $\vec{x}, i, j$, so the 2-skeleton $\text{sk}_2(K_n)$ can be obtained from $\text{sk}_2(X)$ by attaching 2-faces. Since the fundamental group of a complex depends only on its 2-skeleton, we have $\pi_1(\text{sk}_2(X)) = \pi_1(\mathbb{R}^n) = 0$, so $\pi_1(\text{sk}_2(K_n)) = 0$, so $\pi_1(K_n) = 0$. □

The result above can also be understood combinatorially:

Definition 3.4.6. Let K be a simplicial complex. A *path* γ in K is a finite sequence of vertices $v_0, \ldots, v_m$, for some $m > 0$, with v_i, v_{i+1} adjacent for each $i \in \{0, \ldots, m-1\}$. We say that γ *starts at* v_0 and *ends at* v_m, and we may refer to γ as a path *from* v_0 *to* v_m.

Definition 3.4.7. Let K be a simplicial complex, let $u, v, w \in V(K)$, let γ be a path $v_0, \ldots, v_m$ in K from u to v, and let δ be a path $w_0, \ldots, w_n$ in K from v to w. Then we define $\gamma \cdot \delta$ as the path $v_0, \ldots, v_m, w_1, \ldots, w_n$ in K from u to w.

Definition 3.4.8. Let K be a simplicial complex, and let γ, γ' be paths in K. We define a relation $\gamma \simeq \gamma'$ (in words, "γ, γ' are homotopic") inductively:

(1) If $u, v \in V(K)$ are adjacent, then $u, v, u \simeq u$ and $u \simeq u, v, u$.

(2) If $u, v, w \in V(K)$ are contained in a common face $\{u, v, w\}$ of K, then $u, v, w \simeq u, w$ and $u, w \simeq u, v, w$.

(3) If $\gamma \simeq \gamma'$ and $\delta \simeq \delta'$, then $\gamma \cdot \delta \simeq \gamma' \cdot \delta'$.

(4) For any path γ, we have $\gamma \simeq \gamma$.

(5) If $\gamma \simeq \gamma'$ and $\gamma' \simeq \gamma''$, then $\gamma \simeq \gamma''$.

(An induction on the definition shows that $(\simeq)$ is symmetric, and that $\gamma \simeq \gamma'$ only if γ, γ' start at the same vertex and end at the same vertex.)

The usual topological notion of a simplicial complex being simply connected is equivalent to a combinatorial notion using the definitions above (see [17]). As a result, we get the following corollary to Remark 3.4.5:

Corollary 3.4.9. *Let $\vec{u}, \vec{v} \in V(K_n)$, and let γ, γ' be paths in K_n from $\vec{u}$ to $\vec{v}$. Then we have $\gamma \simeq \gamma'$.*

With these preliminaries in hand, we now describe our construction.

3.5 Construction of simplicial complexes from colored k-configurations

We now give a construction producing a labeling of the complex K_n defined in Section 3.4. After taking a quotient of K_n according to this labeling, we obtain a simplicial complex with properties analogous to the 7-vertex triangulation of T^2 (Theorem 3.5.2).

Lemma 3.5.1. *Let $\mathcal{C}$ be a colored $(k+1)$-configuration. Then there exists a surjective labeling $\ell\colon V(K_k) \to P(\mathcal{C})$ of the vertices of K_k, such that:*

(1) If $\vec{u}, \vec{v} \in V(K_k)$ are adjacent, then $\ell(\vec{u}) \neq \ell(\vec{v})$.

(2) Given $\vec{u} \in V(K_k)$ and a point $p \in P(\mathcal{C})$, there is at most one vertex $\vec{v} \in V(K_k)$ adjacent to $\vec{u}$ with $\ell(\vec{v}) = p$. (If $\mathcal{C}$ is also a projective plane, then "at most one" can be replaced with "exactly one.")

(3) If $\vec{u}, \vec{v} \in V(K_k)$ satisfy $\ell(\vec{u}) = \ell(\vec{v})$, then $\ell(\vec{u} + \vec{w}) = \ell(\vec{v} + \vec{w})$ for all $\vec{w} \in V(K_k)$. (That is, ℓ is invariant under translation by $\vec{u} - \vec{v}$.)

Proof. Given adjacent vertices $\vec{u}, \vec{v} \in V(K_k)$ and a label $\ell(\vec{u})$, we will establish a rule for determining $\ell(\vec{v})$. We may uniquely write $\vec{v} = \vec{u} - \vec{e}_i + \vec{e}_j$ for distinct $i, j \in [k+1]$; then our rule proceeds as follows:

- Let l be the line of $\mathcal{C}$ incident with p such that (p, l) has color i in $\mathcal{C}$.
- Let q be the point of $\mathcal{C}$ incident with l such that (q, l) has color j in $\mathcal{C}$.
- Take $\ell(\vec{v}) = q$.

We can also restate the rule using the functions ϕ_c from Definition 3.2.6:

$$\ell(\vec{v}) = (\phi_j \circ \phi_i)(\ell(\vec{u}))$$

(That is, in $G(\mathcal{C})$, following the unique path from the point $\ell(\vec{u})$ along edges with colors i, j, in order, brings us to $\ell(\vec{v})$.)

Now let γ be a path in K_k from $\vec{u}$ to $\vec{v}$, and let $p \in P(\mathcal{C})$. Consider assigning $\ell(\vec{u}) = p$, and then successively applying the rule above on each pair of consecutive vertices of γ, until we obtain a label $\ell(\vec{v}) = q$ at the end of γ. We define $\ell_\gamma(p) = q$. We claim that ℓ_γ respects path homotopy:

Claim. Let γ, γ' be paths in K_k. If $\gamma \simeq \gamma'$, then $\ell_\gamma = \ell_{\gamma'}$.

Proof of claim. We induct on the definition of $(\simeq)$. Cases (3), (4), (5) are clear, so we turn to (1), (2):

Case (1). $\vec{u}, \vec{v} \in V(K_k)$ are adjacent; γ is $\vec{u}, \vec{v}, \vec{u}$, and γ' is $\vec{u}$.

Proof of case (1). Write $\vec{v} = \vec{u} - \vec{e}_i + \vec{e}_j$ as above. Then we have

$$\ell_\gamma = \phi_i \circ \phi_j \circ \phi_j \circ \phi_i = \phi_i \circ \phi_i = 1_{P(\mathcal{C})} = \ell_{\gamma'},$$

where $1_{P(\mathcal{C})}$ is the identity function on $P(\mathcal{C})$.

Case (2). $\vec{u}, \vec{v}, \vec{w} \in V(K_k)$ are pairwise adjacent; γ is $\vec{u}, \vec{v}, \vec{w}$, and γ' is $\vec{u}, \vec{w}$.

Proof of case (2). By Remark 3.4.3, $\vec{u}, \vec{v}, \vec{w}$ are contained in a unique facet F of K_k. On one hand, if F is positive, then we have

$$\vec{u} = \vec{x} + \vec{e}_i, \qquad \vec{v} = \vec{x} + \vec{e}_j, \qquad \vec{w} = \vec{x} + \vec{e}_{k'}.$$

As a result, we have

$$\ell_\gamma = \phi_{k'} \circ \phi_j \circ \phi_j \circ \phi_i = \phi_{k'} \circ \phi_i = \ell_{\gamma'}.$$

On the other hand, if F is negative, then we have

$$\vec{u} = \vec{x} - \vec{e}_i, \qquad \vec{v} = \vec{x} - \vec{e}_j, \qquad \vec{w} = \vec{x} - \vec{e}_{k'}.$$

As a result, we have

$$\ell_\gamma = \phi_j \circ \phi_{k'} \circ \phi_i \circ \phi_j, \qquad \ell_{\gamma'} = \phi_i \circ \phi_{k'}.$$

Since $\phi_i, \phi_j, \phi_{k'}$ are involutions, we have $\ell_{\gamma'}^{-1} = \phi_{k'} \circ \phi_i$, so

$$\ell_\gamma \circ \ell_{\gamma'}^{-1} = \phi_j \circ \phi_{k'} \circ \phi_i \circ \phi_j \circ \phi_{k'} \circ \phi_i = 1_P$$

by the 6-cycle property. Precomposing by $\ell_{\gamma'}$ gives $\ell_\gamma = \ell_{\gamma'}$. This completes the proof of the case and the claim.

We now define ℓ. Fix $\vec{v}_0 \in V(K_k)$ and $p_0 \in P(\mathcal{C})$, and for each $\vec{v} \in V(K_k)$ and each path γ from $\vec{v}_0$ to $\vec{v}$, define $\ell(\vec{v}) = \ell_\gamma(p_0)$. Since K_k is connected, there is at least one such γ. Also, $\ell(\vec{v})$ does not depend on γ, since if γ, γ' are two paths from $\vec{v}_0$ to $\vec{v}$, then $\gamma \simeq \gamma'$ by Corollary 3.4.9, so $\ell_\gamma = \ell_{\gamma'}$ by the claim above. Since $\mathcal{C}$ is connected, ℓ is surjective.

To show that property (1) holds, let γ be a path in K_k from $\vec{v}_0$ to $\vec{u}$, and consider the path $\gamma, \vec{v}$ obtained by appending $\vec{v}$ to γ. Then write $\vec{v} = \vec{u} - \vec{e}_i + \vec{e}_j$ as above, where $i \neq j$. We have

$$\ell(\vec{v}) = \ell_{\gamma,\vec{v}}(p_0) = (\phi_j \circ \phi_i)(\ell_\gamma(p_0)) = (\phi_j \circ \phi_i)(\ell(\vec{u})).$$

It suffices to prove that $(\phi_j \circ \phi_i)(p) \neq p$ for all $p \in P(\mathcal{C})$. This is clear in $G(\mathcal{C})$; if we follow edges of distinct colors i, j, in order, we cannot arrive back at the starting vertex.

To show that property (2) holds, let γ be a path in K_k from $\vec{v}_0$ to $\vec{u}$. The neighbors of $\vec{u}$ in K_k are $\vec{v}_{i,j} = \vec{u} - \vec{e}_i + \vec{e}_j$ for $i, j \in [k+1]$, $i \neq j$. Therefore, $\ell(\vec{v}_{i,j}) = (\phi_j \circ \phi_i)(\ell(\vec{u}))$; equivalently, in $G(\mathcal{C})$, $\ell(\vec{v}_{i,j})$ is obtained from $\ell(\vec{u})$ by following edges of colors i, j, in order. Any i, j with $\ell(\vec{v}_{i,j}) = p$ correspond to a unique line incident with $\ell(\vec{u})$ and p. But $\ell(\vec{u}), p$ lie on at most one common line, so there is at most one vertex $\vec{v}_{i,j}$ with $\ell(\vec{v}_{i,j}) = p$. (For a projective plane $\mathcal{C}$, replace "at most" with "exactly.")

To show that property (3) holds, first suppose $\vec{w}$ is adjacent to the zero vector, say $\vec{w} = -\vec{e}_i + \vec{e}_j$. Let γ be a path in K_k from $\vec{v}_0$ to $\vec{u}$. Then

$$\ell(\vec{u} + \vec{w}) = \ell_{\gamma,\vec{w}}(p_0) = (\phi_j \circ \phi_i)(\ell_\gamma(p_0)) = (\phi_j \circ \phi_i)(\ell(\vec{u})).$$

Hence if $\ell(\vec{u}) = \ell(\vec{v})$, then $\ell(\vec{u} + \vec{w}) = \ell(\vec{v} + \vec{w})$. We obtain property (3) for all $\vec{w}$ by induction on the distance $d(\vec{0}, \vec{w})$ in K_k. This completes the proof. $\square$

Theorem 3.5.2. *Let $\mathcal{C}$ be a colored $(k+1)$-configuration with n points and n lines. Then there exists a connected k-dimensional simplicial complex $K(\mathcal{C})$ with $\pi_1(K(\mathcal{C})) \cong \mathbb{Z}^k$, such that:*

(1) $K(\mathcal{C})$ has exactly n vertices.

(2) $K(\mathcal{C})$ contains a copy of $\mathcal{C}$, consisting of n facets of K, and a copy of the dual $(k+1)$-configuration $\mathcal{C}^*$, consisting of n other facets of K. These two copies fully describe K, in that these $2n$ facets are all the facets of K, and every face of K is contained in a facet.

(3) Suppose G acts on $\mathcal{C}$ via a group action ρ. Then G acts on $K(\mathcal{C})$ via ρ_P, the action induced by ρ on the points of $\mathcal{C}$. If ρ is free, then ρ_P is free also.

(4) If $\mathcal{C}$ is also a projective plane, then $K(\mathcal{C})$ is 2-neighborly.

Proof. Apply Lemma 3.5.1 to obtain a labeling $\ell \colon V(K_k) \to P(\mathcal{C})$. Then define

$$H = \{\vec{v} \in A_k : \ell(\vec{v}) = \ell(\vec{0})\}.$$

For $\vec{u}, \vec{v} \in A_k$, with $\ell(\vec{u}), \ell(\vec{v}) = 0$, property (3) of Lemma 3.5.1 implies

$$\ell(\vec{u} + \vec{v}) = \ell(\vec{0} + \vec{v}) = \ell(\vec{v}) = \ell(\vec{0}).$$

Likewise, property (3) also implies

$$\ell(-\vec{v}) = \ell(\vec{0} - \vec{v}) = \ell(\vec{v} - \vec{v}) = \ell(\vec{0}).$$

Therefore, H is an additive subgroup of A_k. Moreover, for any two $\vec{u}, \vec{v} \in A_k$, we have the following chain of equivalences:

$$\ell(\vec{u}) = \ell(\vec{v}) \quad \Leftrightarrow \quad \ell(\vec{u} - \vec{v}) = \ell(\vec{0}) \quad \Leftrightarrow \quad \vec{u} - \vec{v} \in H.$$

Hence the orbits of H in A_k correspond bijectively to points $p \in P(\mathcal{C})$ in the image of ℓ. Since ℓ is surjective, there are exactly n such orbits.

We claim that H spans the vector space $\{(x_1, \ldots, x_{k+1}) \in \mathbb{R}^{k+1} : \sum_i x_i = 0\}$. Suppose otherwise; then there exists $\vec{v} \in A_k$ with $\vec{v} \notin \text{span } H$. Then we also have $\lambda\vec{v} \notin \text{span } H$ for all nonzero $\lambda \in \mathbb{Z}$. But then $\ell(\lambda\vec{v})$ is distinct for distinct $\lambda \in \mathbb{Z}$, a contradiction since $P(\mathcal{C})$ is finite. Therefore, H is a k-dimensional lattice, so we have $H \cong \mathbb{Z}^k$.

By Remark 3.4.4, A_k acts on K_k by addition, so H does also. Therefore, the quotient K_k/H is a well-defined k-dimensional CW complex, and in fact is a simplicial complex by (1), (2) of Lemma 3.5.1. Since H is a "covering space action" in the sense of Hatcher [9], and K_k is simply connected, we have $\pi_1(K_k/H) \cong H \cong \mathbb{Z}^k$ (See [9], Proposition 1.40). Hence we may take $K(\mathcal{C}) = K_k/H$.

It remains to show properties (1) through (4) from the theorem statement. Note that property (1) holds since there are n orbits of H in A_k, and property (4) holds by (2) of Lemma 3.5.1.

To show that property (2) holds, consider two adjacent vectors $\vec{u}, \vec{v} \in K_k$, and consider the line l to determine $\ell(\vec{v})$ from $\ell(\vec{u})$, or vice versa. If $\vec{u} = \vec{x} + \vec{e}_i$, $\vec{v} = \vec{x} + \vec{e}_j$, then $l = \phi_i(\ell(\vec{x} + \vec{e}_i)) = \phi_j(\ell(\vec{x} + \vec{e}_j))$. Hence for the positive facet rooted at $\vec{x}$, the line l is shared by all pairs of vertices in the facet. Therefore, the positive facets of $K(\mathcal{C})$ correspond to the lines of $\mathcal{C}$.

To show that property (3) holds, it suffices to prove that ρ_P maps facets of $K(\mathcal{C})$ to other facets. This holds for positive facets, since these correspond to lines of $\mathcal{C}$ by (2). Since ρ respects colors, ρ_P also respects the orientation of positive facets. Since each edge of $K(\mathcal{C})$ is contained in a positive facet, ρ_P represents a translation in K_k. Hence ρ_P also respects negative facets. □

3.6 Construction of colored k-configurations from planar difference sets

We begin by defining planar difference sets (see [1]):

Definition 3.6.1. Let G be an abelian group. A *planar difference set* in G is a subset $A \subseteq G$, such that for each $g \in G$ other than the identity, there exist unique $a_1, a_2 \in A$ with $g = a_1 - a_2$. The *order* of A is $|A| - 1$.

Note that $|G| = (k+1) \cdot k + 1 = k^2 + k + 1$. In fact, a planar difference set forms a projective plane, in the following way:

Remark 3.6.2 (Singer [19]). *Let G be an abelian group, and let A be a planar difference set in G, with $|A| = k + 1$. Then we obtain a projective plane of order k, where:*

- *The point set P is the set of elements of G.*
- *The line set L is also the set of elements of G.*
- *A point $p \in P$ and line $l \in L$ are incident if $p - l \in A$.*

Proof. Fix distinct points $p_1, p_2 \in G$. Then a line $l \in G$ is incident with both of p_1, p_2 if and only if $p_1 - l = a_1$ and $p_2 - l = a_2$ for $a_1, a_2 \in A$, implying $p_1 - p_2 = a_1 - a_2$. Such $a_1, a_2 \in A$ are uniquely determined by p_1, p_2, so any two points lie on exactly one common line. The dual statement holds similarly. □

For example, the projective planes $PG(2, \mathbb{F}_q)$ all arise in this way:

Remark 3.6.3 (Singer [19]). *Let q be a prime power. Then the finite projective plane $PG(2, \mathbb{F}_q)$ corresponds to a planar difference set in $\mathbb{Z}_{q^2+q+1}$.*

However, it is not known whether all finite projective planes arising from planar difference sets are isomorphic to $PG(2, \mathbb{F}_q)$; see [10] for a partial result in this direction. We now give our construction.

Theorem 3.6.4. *Let G be an abelian group, and let A be a planar difference set in G with $|A| = k + 1$. The corresponding projective plane can be colored, giving a colored $(k + 1)$-configuration $\mathcal{C}$. Moreover, G acts freely on $\mathcal{C}$.*

Proof. Let $\mathcal{C}$ be the projective plane described in Remark 3.6.2. Then $\mathcal{C}$ is connected by Remark 3.2.4, since any two points lie on a common line.

We define a coloring $\chi\colon R(\mathcal{C}) \to A$ by $\chi(p, l) = p - l$; note that $p - l \in A$ since p, l are incident in $\mathcal{C}$. To check the 6-cycle property, let $a, b, c \in A$, and consider the path $p_0, l_0, p_1, l_1, p_2, l_2, p_3$ in $G(\mathcal{C})$ with

$$\begin{aligned} p_0 - l_0 &= a \\ p_1 - l_0 &= b \\ p_1 - l_1 &= c \\ p_2 - l_1 &= a \\ p_2 - l_2 &= b \\ p_3 - l_2 &= c \end{aligned}$$

Taking an alternating sum gives $p_0 - p_3 = 0$, so the 6-cycle property holds.

The free group action of G on $\mathcal{C}$ is given by $g \cdot p = g + p$ for points $p \in G$, and $g \cdot l = g + l$ for lines $l \in G$; this preserves incidences and colors. □

Corollary 3.6.5. *Let q be a prime power. Then $PG(2, \mathbb{F}_q)$ can be colored, giving a colored $(q + 1)$-configuration $\mathcal{C}$ with $q^2 + q + 1$ points and $q^2 + q + 1$ lines. Moreover, $\mathbb{Z}_{q^2+q+1}$ acts freely on $\mathcal{C}$.*

Proof. By Remark 3.6.3 and Theorem 3.6.4. □

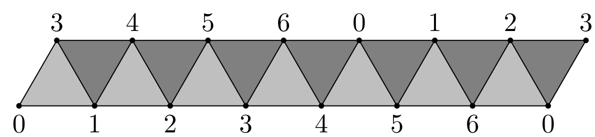
As a corollary, we get the following promised result:

Corollary 3.6.6. *Let q be a prime power. Then there exists a connected q-dimensional simplicial complex K with $\pi_1(K) \cong \mathbb{Z}^q$, such that:*

- *K has exactly $q^2 + q + 1$ vertices.*
- *K contains two copies of $PG(2, \mathbb{F}_q)$, each consisting of $q^2 + q + 1$ facets of K. These two copies fully describe K, in that these $2(q^2 + q + 1)$ facets are all of the facets of K, and every face of K is contained in a facet.*
- *K is cyclic; the cyclic group $\mathbb{Z}_{q^2+q+1}$ acts freely on K.*
- *K is 2-neighborly.*

Proof. By Corollary 3.6.5 and Theorem 3.5.2, since $PG(2, \mathbb{F}_q)$ is self-dual. □

If we express $PG(2, \mathbb{F}_q)$ as a planar difference set, the duality between positive and negative facets becomes clear. For example, $PG(2, \mathbb{F}_2)$ corresponds to the planar difference set $\{0, 1, 3\}$ in $\mathbb{Z}/7\mathbb{Z}$. The positive facets are translates of $\{0, 1, 3\}$, and the negative facets are translates of $\{0, -1, -3\}$:



3.7 Construction of colored k-configurations from commutative semifields

In this section, we give a construction of colored k-configurations from commutative semifields. A semifield has the properties of a field, except that multiplication is not required to be associative or commutative. More concretely:

Definition 3.7.1 ([12]; see [22])**.** A *semifield* consists of a set S and operations $(+), (\cdot)$, such that:

- S forms a group under addition; we call the additive identity 0.
- If $a, b \in S$, $a \neq 0$, then there exists unique $x \in S$ with $a \cdot x = b$.
- If $a, b \in S$, $a \neq 0$, then there exists unique $y \in S$ with $y \cdot a = b$.
- If $a, b, c \in S$, then $a \cdot (b + c) = ab + ac$, and $(a + b) \cdot c = ac + bc$.
- There exists $1 \in S$ with $1 \neq 0$, and $1 \cdot a = a \cdot 1 = a$ for all $a \in S$.

This definition is important in the classification of projective planes. Any projective plane admits a coordinatization with coordinates in a *ternary ring*; conversely, any ternary ring gives rise to a projective plane. If the ternary ring is also a semifield, the corresponding projective plane has certain symmetries.

Moreover, if the ternary ring is also a field, the corresponding projective plane is $PG(2, \mathbb{F}_q)$. (See [22] for definitions and details.)

Like fields, the number of elements in a semifield is always a prime power (see [12], Section 2.5). But there are commutative semifields that are not fields; see [22] for examples. We outline the construction of projective planes from semifields; this is a special case of the general construction for ternary rings:

Theorem 3.7.2 (Hall [8])**.** *Let F be a semifield with $|F| = q$. Then there exists a finite projective plane of order q.*

Proof. The point set P consists of:

- Points (x, y) for $x, y \in F$.
- Points (x) for $x \in F \sqcup \{\infty\}$; this is a "point at infinity" for slope x.

The line set L consists of:

- Lines $[a, b]$ for $a, b \in F$; this is the "line" $y = ax + b$.
- Lines $[a]$ for $a \in F \sqcup \{\infty\}$; this is the "line" $x = a$.

The set of incidence relations R consists of:

- Incidences $((x, y), [a, b])$ for $x, y, a, b \in F$ with $y = ax + b$.
- Incidences $((x, y), [x])$ for $x, y \in F$.
- Incidences $((a), [a, b])$ for $a, b \in F$.
- Incidences $((x), [\infty])$ for $x \in F$.
- Incidences $((\infty), [a])$ for $a \in F$.
- The incidence $((\infty), [\infty])$.

We omit the proofs of the projective plane properties. □

We now give our construction.

Theorem 3.7.3. *Let F be a commutative semifield with $|F| = q$. Then there exists a colored q-configuration with q^2 points and q^2 lines.*

Proof. Starting with the finite projective plane from Theorem 3.7.2, delete all points and lines other than those of the form $(x, y), [a, b]$. This deletes a line through each remaining point and a point on each remaining line; hence the result is a q-configuration $\mathcal{C}$ with q^2 points and q^2 lines.

To show that $\mathcal{C}$ is connected, it suffices by Remark 3.2.4 to show that there exists a path in $G(\mathcal{C})$ between any two points $(x_1, y_1), (x_2, y_2)$. If $x_1 \neq x_2$, then let $a \in F$ satisfy $a \cdot (x_2 - x_1) = y_2 - y_1$, and let $b = y_1 - a \cdot x_1$. Then (x_1, y_1) lies on $[a, b]$ by the definition of b, and (x_2, y_2) lies on $[a, b]$ by:

$$\begin{aligned} a \cdot x_2 + b &= a \cdot x_2 + (y_1 - a \cdot x_1) \\ &= a \cdot (x_2 - x_1) + y_1 \\ &= (y_2 - y_1) + y_1 \\ &= y_2. \end{aligned}$$

Hence $(x_1, y_1), (x_2, y_2)$ are adjacent if $x_1 \neq x_2$. Since F has at least two distinct elements $0, 1$, there exists a path in $G(\mathcal{C})$ between any two points of $\mathcal{C}$.

Define a coloring $\chi\colon R(\mathcal{C}) \to F$ by $\chi((x, y), [a, b]) = x + a$. In an incidence $((x, y), [a, b])$, each of y, b are uniquely determined by the other three variables, which implies that χ is a q-edge coloring of $G(\mathcal{C})$. To check the 6-cycle property, let $c, d, e \in F$, and consider the path (x_0, y_0), $[a_1, b_1]$, (x_1, y_1), $[a_2, b_2]$, (x_2, y_2), $[a_3, b_3]$, (x_3, y_3) in $G(\mathcal{C})$ with

$$\begin{aligned} x_0 + a_1 &= c \\ x_1 + a_1 &= d \\ x_1 + a_2 &= e \\ x_2 + a_2 &= c \\ x_2 + a_3 &= d \\ x_3 + a_3 &= e \end{aligned}$$

Taking an alternating sum gives $x_0 - x_3 = 0$; it remains to show $y_0 = y_3$. Note that $a_1 = c - x_0$, $a_2 = c - d + e - x_0$, and $a_3 = e - x_0$. We have

$$\begin{aligned} y_3 - y_0 &= (y_3 - y_2) + (y_2 - y_1) + (y_1 - y_0) \\ &= a_3 \cdot (x_3 - x_2) + a_2 \cdot (x_2 - x_1) + a_1 \cdot (x_1 - x_0) \\ &= (e - x_0) \cdot (e - d) + (c - d + e - x_0) \cdot (c - e) + (c - x_0) \cdot (d - c) \\ &= c \cdot d + d \cdot e + e \cdot c - d \cdot c - e \cdot d - c \cdot e \\ &= 0. \end{aligned}$$

where we use commutativity in the last step. This completes the proof. □

In Theorem 3.7.3, we can take F to be the field $\mathbb{F}_q$, which gives a colored q-configuration $\mathcal{C}$; contrast this with Corollary 3.6.5, which gives a colored $(q+1)$-configuration $\mathcal{C}'$. The underlying q-configuration of $\mathcal{C}$ can be completed to that of $\mathcal{C}'$ by adding points and lines at infinity, but the coloring of $\mathcal{C}$ cannot be extended to $\mathcal{C}'$; in this sense, the two constructions are distinct.

3.8 Relationships with Sidon sets & linear codes

We now explain the connection between Sidon sets, linear codes, and colored k-configurations. To begin, we define a Sidon set:

Definition 3.8.1 ([18]; see [14]). Let G be an abelian group, written additively. A set $B = \{b_0, b_1, \ldots, b_k\} \subseteq G$ is a *Sidon set of order* h if the sums $b_{i_1} + \cdots + b_{i_h}$, $0 \le i_1 \le \cdots \le i_h \le k$, are all different.

For $h = 2$, this condition is equivalent to the condition that the differences $b_i - b_j$ for $i \ne j$ are all distinct, so $|G| \ge k^2 + k + 1$. Then a planar difference set in G is exactly a Sidon set B of order 2 with $|B| = k$ and $|G| = k^2 + k + 1$. As with planar difference sets, Sidon sets may be translated; if B is a Sidon set of order h in G, then so is $\{b + g : b \in B\}$ for any $g \in G$.

Next, we define a linear code:

Definition 3.8.2 (see [14]). A *linear code* with radius r in A_k is a lattice $\mathcal{L} \subseteq A_k$, such that the balls $B_r(\vec{x}) = \{\vec{y} : d(\vec{x}, \vec{y}) \le r\}$ for $\vec{x} \in \mathcal{L}$ are all disjoint. We say that $\mathcal{L}$ is *perfect* if the balls $B_r(\vec{x})$ also cover A_k.

Then we have a correspondence between Sidon sets and linear codes; recall from Section 3.4 that $A_n \cong \mathbb{Z}^n$ on ignoring the last coordinate:

Theorem 3.8.3 (Kovačević [14]).

(a) Let $B = \{0, b_1, \ldots, b_k\}$ be a Sidon set of order 2 in an abelian group G, and suppose B generates G. Then the lattice

$$\mathcal{L} = \{\vec{x} \in A_k : \sum_{i=1}^{k} x_i b_i = 0\}$$

is a linear code with radius 1 in A_k, and $G \cong A_k/\mathcal{L}$. (Here $x_i b_i$ denotes the sum in G of $|x_i|$ copies of b_i if $x_i > 0$ and of $-b_i$ if $x_i < 0$.)

(b) Conversely, if $\mathcal{L}$ is a linear code with radius 1 in A_k, then the group $G = A_k/\mathcal{L}$ contains a Sidon set B of order 2 with $|B| = k + 1$, such that B generates G.

We also have a correspondence with colored $(k+1)$-configurations:

Theorem 3.8.4. *We have one-to-one correspondences (up to isomorphism) between any two of the following three mathematical structures:*

(1) Pairs (G, B), where G is an abelian group with $|G| = n$, and B is a Sidon set of order 2 in G with $|B| = k + 1$.

(2) Linear codes $\mathcal{L}$ with radius 1 in A_k, with $|A_k/\mathcal{L}| = n$.

(3) Colored $(k+1)$-configurations $\mathcal{C}$ with n points and n lines.

Proof. Theorem 3.8.3 gives a correspondence between (1), (2), and Lemma 3.5.1 gives a map from (3) to (2), since if $\ell(\vec{0}) = p$, then $\ell^{-1}(p)$ is a linear code with radius 1 by the properties of the lemma. We now give a map from (2) to (3).

Let $\mathcal{L}$ be a linear code with radius 1 in A_k, with $|A_k/\mathcal{L}| = n$. We define a colored $(k+1)$-configuration $\mathcal{C}$ as follows:

- The point set P is the set of elements of $A_k/\mathcal{L}$.
- The line set L is also the set of elements of $A_k/\mathcal{L}$.
- A point $p \in P$ and line $l \in L$ are incident if $p - l = \vec{e}_i - \vec{e}_{k+1}$ for some $i \in [k+1]$; the color of the incidence is i.

Fix distinct points $p_1, p_2 \in A_k/\mathcal{L}$. Then a line $l \in A_k/\mathcal{L}$ is incident with both of p_1, p_2 if and only if $p_1 - l = \vec{e}_i - \vec{e}_{k+1}$ and $p_2 - l = \vec{e}_j - \vec{e}_{k+1}$ for $i, j \in [k+1]$, implying $p_1 - p_2 = \vec{e}_i - \vec{e}_j$, and $i \neq j$. Now the vectors $\vec{e}_i - \vec{e}_j$ for distinct $i, j \in [k+1]$ have pairwise distance 1 or 2 in A_k, and hence are pairwise distinct in $A_k/\mathcal{L}$. Therefore, there is at most one choice of i, j with $p_1 - p_2 = \vec{e}_i - \vec{e}_j$, so there is at most one l incident with both of p_1, p_2. The dual statement holds similarly, so $\mathcal{C}$ is a $(k+1)$-configuration.

To show that $\mathcal{C}$ is connected, it suffices by Remark 3.2.4 to show that there exists a path in $G(\mathcal{C})$ between any two points $p_1, p_2 \in A_k/\mathcal{L}$. Suppose we have $p_1 - p_2 = \vec{e}_i - \vec{e}_{k+1}$; then p_1, p_2 both lie on the line p_2. The general case follows by writing $p_1 - p_2$ as an integer combination of vectors of the form $\vec{e}_i - \vec{e}_{k+1}$.

To show that $\mathcal{C}$ has the 6-cycle property, let $a, b, c \in [k+1]$, and consider the path $p_0, l_0, p_1, l_1, p_2, l_2, p_3$ in $G(\mathcal{C})$ with

$$\begin{aligned}
p_0 - l_0 &= \vec{e}_a - \vec{e}_{k+1} \\
p_1 - l_0 &= \vec{e}_b - \vec{e}_{k+1} \\
p_1 - l_1 &= \vec{e}_c - \vec{e}_{k+1} \\
p_2 - l_1 &= \vec{e}_a - \vec{e}_{k+1} \\
p_2 - l_2 &= \vec{e}_b - \vec{e}_{k+1} \\
p_3 - l_2 &= \vec{e}_c - \vec{e}_{k+1}
\end{aligned}$$

Taking an alternating sum gives $p_0 - p_3 = 0$, so the 6-cycle property holds.

Hence we have a construction η from (2) to (3) and a construction θ from (3) to (2). It remains to show that η, θ are inverses.

Claim. Let $\mathcal{L}$ be a linear code with radius 1 in A_k, $\mathcal{C} = \eta(\mathcal{L})$, $\mathcal{L}' = \theta(\mathcal{C})$, where θ constructs $\ell\colon A_k \to A_k/\mathcal{L}$ starting with $\ell(\vec{0}) = \vec{0}$. Then we have $\mathcal{L} = \mathcal{L}'$.

Proof of claim. It suffices to prove $\ell(\vec{x}) = \vec{x}$ for all $\vec{x} \in A_k$. For points $p \in A_k/\mathcal{L}$, we have $\phi_i(p) = p - \vec{e}_i + \vec{e}_{k+1}$; for lines $l \in A_k/\mathcal{L}$, we have $\phi_i(l) = p + \vec{e}_i - \vec{e}_{k+1}$. Therefore, if $\ell(\vec{u}) = \vec{u}$ for $\vec{u} \in A_k$, then

$$\begin{aligned}
\ell(\vec{u} - \vec{e}_i + \vec{e}_j) &= (\phi_j \circ \phi_i)(\ell(\vec{u})) \\
&= (\phi_j \circ \phi_i)(\vec{u}) \\
&= \vec{u} - \vec{e}_i + \vec{e}_j.
\end{aligned}$$

Then by induction, $\ell(\vec{x}) = \vec{x}$ for all $\vec{x} \in A_k$ as desired.

Claim. Let $\mathcal{C}$ be a colored $(k+1)$-configuration, $\mathcal{L} = \theta(\mathcal{C})$, $\mathcal{C}' = \eta(\mathcal{L})$. Then we have an isomorphism $\mathcal{C} \cong \mathcal{C}'$.

Proof of claim. Let $\ell\colon A_k \to P(\mathcal{C})$ be the labeling constructed by θ, and define $\ell^{-1}\colon P(\mathcal{C}) \to A_k/\mathcal{L}$ such that $\ell \circ \ell^{-1}$ is the identity map on $P(\mathcal{C})$, and $\ell^{-1} \circ \ell$ is the quotient map $A_k \to A_k/\mathcal{L}$.

We claim $\ell^{-1}(\phi_i(l)) = \ell^{-1}(\phi_j(l)) + \vec{e}_i - \vec{e}_j$, for all $i, j \in [k+1]$. To see this, take the identity $\ell(\vec{x} - \vec{e}_i + \vec{e}_j) = (\phi_j \circ \phi_i)(\ell(\vec{x}))$, and let

$$l = \phi_j(\ell(\vec{x} - \vec{e}_i + \vec{e}_j)) = \phi_i(\ell(\vec{x})).$$

Then we have $\vec{x} = \ell^{-1}(\phi_j(l)) + \vec{e}_i - \vec{e}_j$, and $\vec{x} = \ell^{-1}(\phi_i(l))$. Therefore, we have

$$\ell^{-1}(\phi_i(l)) = \ell^{-1}(\phi_j(l)) + \vec{e}_i - \vec{e}_j.$$

Now we define a map $\mathcal{C} \to \mathcal{C}'$:

- Map $p \in P(\mathcal{C})$ to $\ell^{-1}(p)$ in $P(\mathcal{C}') = A_k/\mathcal{L}$.
- Map $l \in L(\mathcal{C})$ to $\ell^{-1}(\phi_{k+1}(l))$ in $L(\mathcal{C}') = A_k/\mathcal{L}$.

We have the following chain of logical equivalences:

$$\begin{aligned} &p, l \text{ incident in } \mathcal{C} \\ &\quad\Leftrightarrow\quad p = \phi_i(l), && \text{some } i \in [k+1] \\ &\quad\Leftrightarrow\quad \ell^{-1}(p) = \ell^{-1}(\phi_i(l)), && \text{some } i \in [k+1] \\ &\quad\Leftrightarrow\quad \ell^{-1}(p) = \ell^{-1}(\phi_{k+1}(l)) + \vec{e}_i - \vec{e}_{k+1}, && \text{some } i \in [k+1] \\ &\quad\Leftrightarrow\quad \ell^{-1}(p), \ell^{-1}(\phi_{k+1}(l)) \text{ incident in } \mathcal{C}' \end{aligned}$$

Hence our map $\mathcal{C} \to \mathcal{C}'$ preserves both incidences and colors, which are given by the index i above, so $\mathcal{C} \cong \mathcal{C}'$ as desired.

By the claims above, η, θ are inverses, which completes the proof. □

Let $\mathcal{C}$ be the colored $(q+1)$-configuration obtained from a planar difference set via Theorem 3.6.4. Then in the theorem above, we have $G \cong \mathbb{Z}_{q^2+q+1}$, $|B| = q+1$, and $\mathcal{L}$ perfect. This case of the correspondence between (1) and (2) is discussed in Section 3 of [14]; in particular, we recover a result of Singer on Sidon sets ([19]; see [3]). We also obtain a converse to Theorem 3.6.4:

Corollary 3.8.5. *Let $\mathcal{C}$ be a colored $(k+1)$-configuration, such that $\mathcal{C}$ is also a projective plane. Then $\mathcal{C}$ arises from a planar difference set (via Theorem 3.6.4).*

Proof. Since $\mathcal{C}$ is a projective plane, $\mathcal{C}$ has k^2+k+1 points and k^2+k+1 lines. Apply Theorem 3.8.4 to obtain a pair (G, B), where G is an abelian group with $|G| = k^2+k+1$, and B is a Sidon set of order 2 in G with $|B| = k+1$. Then B is a planar difference set as discussed above. Apply Theorem 3.8.4 in reverse to obtain a colored $(k+1)$-configuration $\mathcal{C}'$ with $\mathcal{C} \cong \mathcal{C}'$. The map from (1) to (3) in Theorem 3.8.4 is equivalent to the construction in Theorem 3.6.4, so $\mathcal{C}'$ arises from the planar difference set B in G. □

Now let $\mathcal{C}$ be the colored q-configuration obtained from a commutative semifield F with $|F| = q$ via Theorem 3.7.3. Then in Theorem 3.8.4, we have $|G| = q^2$, $|B| = q$. This can be compared with the following known result:

Remark 3.8.6 (Bose [2]; see [3, 14]). *Let q be a prime power. Then there exists a Sidon set B of order 2 in $\mathbb{Z}_{q^2-1}$, with $|B| = q$.*

Our Sidon set with $|G| = q^2$, $|B| = q$, is suboptimal in the sense that G is larger than necessary, but we are not aware that it exists in the literature. We close with several questions for further research:

Question 3.8.7. *Does the Sidon set in Remark 3.8.6 have a nice description as a colored q-configuration?*

Question 3.8.8. *Which connected k-configurations $\mathcal{C}$ can be colored to form a colored k-configuration?*

Note that if $\mathcal{C}$ is also a projective plane, then $\mathcal{C}$ can be colored if and only if $\mathcal{C}$ arises from a planar difference set, by Theorem 3.6.4 and Corollary 3.8.5. The question still stands for $\mathcal{C}$ not a projective plane.

References

[1] Thomas Beth, Dieter Jungnickel, and Hanfried Lenz. *Design theory. Vol. I*. Second. Vol. 69. Encyclopedia of Mathematics and its Applications. Cambridge University Press, Cambridge, 1999, pp. xx+1100.

[2] R. C. Bose. “An affine analogue of Singer’s theorem”. In: *J. Indian Math. Soc. (N.S.)* 6 (1942), pp. 1–15.

[3] R. C. Bose and S. Chowla. “Theorems in the additive theory of numbers”. In: *Comment. Math. Helv.* 37 (1962/63), pp. 141–147.

[4] R. H. Bruck and H. J. Ryser. “The nonexistence of certain finite projective planes”. In: *Canad. J. Math.* 1 (1949), pp. 88–93.

[5] J. H. Conway and N. J. A. Sloane. *Sphere packings, lattices and groups*. Third. Vol. 290. Grundlehren der Mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences]. With additional contributions by E. Bannai, R. E. Borcherds, J. Leech, S. P. Norton, A. M. Odlyzko, R. A. Parker, L. Queen and B. B. Venkov. Springer-Verlag, New York, 1999, pp. lxxiv+703.

[6] H. S. M. Coxeter. “The derivation of Schoenberg’s star-polytopes from Schoute’s simplex nets”. In: *The geometric vein*. Springer, New York-Berlin, 1981, pp. 149–164.

[7] Branko Grünbaum. *Configurations of points and lines*. Vol. 103. Graduate Studies in Mathematics. American Mathematical Society, Providence, RI, 2009, pp. xiv+399.

[8] Marshall Hall. “Projective planes”. In: *Trans. Amer. Math. Soc.* 54 (1943), pp. 229–277.

[9] Allen Hatcher. *Algebraic topology*. Cambridge University Press, Cambridge, 2002, pp. xii+544.

[10] Yiwei Huang and Bernhard Schmidt. “Uniqueness of some cyclic projective planes”. In: *Des. Codes Cryptogr.* 50.2 (2009), pp. 253–266.

[11] M. Jungerman and G. Ringel. “Minimal triangulations on orientable surfaces”. In: *Acta Math.* 145.1-2 (1980), pp. 121–154.

[12] Donald Ervin Knuth. *FINITE SEMIFIELDS AND PROJECTIVE PLANES*. Thesis (Ph.D.)–California Institute of Technology. ProQuest LLC, Ann Arbor, MI, 1963, (no paging).

[13] Dénes König. “Über Graphen und ihre Anwendung auf Determinantentheorie und Mengenlehre”. In: *Math. Ann.* 77.4 (1916), pp. 453–465.

[14] Mladen Kovačević. *Sidon Sets, Difference Sets, and Codes in A_n Lattices*. 2019. arXiv: `1409.5276 [math.CO]`.

[15] W. Kühnel and G. Lassmann. “Combinatorial d-tori with a large symmetry group”. In: *Discrete Comput. Geom.* 3.2 (1988), pp. 169–176.

[16] Frank H. Lutz. *Triangulated Manifolds with Few Vertices: Combinatorial Manifolds*. 2005. eprint: `math/0506372`.

[17] Eric Reynaud. “Algebraic fundamental group and simplicial complexes”. In: *J. Pure Appl. Algebra* 177.2 (2003), pp. 203–214.

[18] S. Sidon. “Ein Satz über trigonometrische Polynome und seine Anwendung in der Theorie der Fourier-Reihen”. In: *Math. Ann.* 106.1 (1932), pp. 536–539.

[19] James Singer. “A theorem in finite projective geometry and some applications to number theory”. In: *Trans. Amer. Math. Soc.* 43.3 (1938), pp. 377–385.

[20] O. Veblen and J. H. Maclagan-Wedderburn. “Non-Desarguesian and non-Pascalian geometries”. In: *Trans. Amer. Math. Soc.* 8.3 (1907), pp. 379–388.

[21] L. Vietoris. “Über den höheren Zusammenhang kompakter Räume und eine Klasse von zusammenhangstreuen Abbildungen”. In: *Math. Ann.* 97.1 (1927), pp. 454–472.

[22] Charles Weibel. “Survey of non-Desarguesian planes”. In: *Notices of the AMS* 54.10 (2007), pp. 1294–1303.

Chapter 4

Clean tangled clutters, simplices, and projective geometries

Joint work with Ahmad Abdi and Gérard Cornuéjols.

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Abstract

A clutter is *clean* if it has no delta or the blocker of an extended odd hole minor, and it is *tangled* if its covering number is two and every element appears in a minimum cover. Clean tangled clutters have been instrumental in progress towards several open problems on ideal clutters, including the $\tau = 2$ Conjecture.

Let $\mathcal{C}$ be a clean tangled clutter. It was recently proved that $\mathcal{C}$ has a fractional packing of value two. Collecting the supports of all such fractional packings, we obtain what is called the *core* of $\mathcal{C}$. The core is a duplication of the cuboid of a set of $0-1$ points, called the *setcore* of $\mathcal{C}$.

In this paper, we prove three results about the setcore. First, the convex hull of the setcore is a full-dimensional polytope containing the center point of the hypercube in its interior. Secondly, this polytope is a simplex if, and only if, the setcore is the cocycle space of a projective geometry over the two-element field. Finally, if this polytope is a simplex of dimension more than three, then $\mathcal{C}$ has the clutter of the lines of the Fano plane as a minor.

Our results expose a fascinating interplay between the combinatorics and the geometry of clean tangled clutters.

4.1 Introduction

A *clutter* is a family $\mathcal{C}$ of subsets of a finite set V where no set contains another one [15]. We refer to V as the *ground set*, to the elements in V simply as *elements*, and to the sets in $\mathcal{C}$ as *members*. A *transversal* is any subset of V that intersects every member exactly once, whereas a *cover* is any subset of V that intersects every member at least once. A cover is *minimal* if it does not contain another cover. The family of the minimal covers of $\mathcal{C}$ forms another clutter over ground set V; this clutter is called the *blocker of* $\mathcal{C}$ and is denoted $b(\mathcal{C})$. It is well-known that $b(b(\mathcal{C})) = \mathcal{C}$ [15, 17]. Given disjoint $I, J \subseteq V$, the *minor of* $\mathcal{C}$ obtained after *deleting* I and *contracting* J is the clutter $\mathcal{C} \setminus I/J$ over ground set $V - (I \cup J)$ whose members are the inclusion-wise sets in $\{C - J : C \in \mathcal{C}, C \cap I = \emptyset\}$. It is well-known that $b(\mathcal{C} \setminus I/J) = b(\mathcal{C})/I \setminus J$ [22].

A *delta* is any clutter over a ground set of cardinality at least three, say $\{a_1, a_2, a_3, \ldots, a_n\}$, whose members are

$$\{a_2, a_3, \ldots, a_n\} \qquad \text{and} \qquad \{a_1, a_i\}, i = 2, \ldots, n.$$

(See Figure 4.1.) Observe that a delta is *identically self-blocking*, that is, every member is a minimal cover, and vice versa. Observe further that the elements and members of a delta correspond to the points and lines of a degenerate projective plane.

An *extended odd hole* is any clutter over a ground set of cardinality at least five and odd, say $\{a_1, \ldots, a_n\}$, whose minimum cardinality members are $\{a_1, a_2\}, \{a_2, a_3\}, \ldots, \{a_{n-1}, a_n\}, \{a_n, a_1\}$. That is, the minimum cardinality members correspond to the edges of an odd hole. (See Figure 4.1.) Note that there may exist other members, but those members would have cardinality at least three. Observe that every cover of an extended odd hole with n elements has cardinality at least $\frac{n+1}{2}$.

Definition 4.1.1 ([9]). A clutter is *clean* if it has no minor that is a delta or the blocker of an extended odd hole.

Observe that if a clutter is clean, then so is every minor of it. Clean clutters were introduced recently and a polynomial recognition algorithm was provided for them [4]. The class of clean clutters includes *ideal* clutters, clutters without an *intersecting minor*, and *binary* clutters [9].

The *covering number* of a clutter $\mathcal{C}$, denoted $\tau(\mathcal{C})$, is the minimum cardinality of a cover.

Definition 4.1.2 ([11]). A clutter is *tangled* if it has covering number two and every element belongs to a minimum cover.

Observe that if a clutter has covering number at least two, then it has a tangled minor, obtained by repeatedly deleting elements that keep the covering number at least two.

Let us define an important class of tangled clutters. A clutter is a *cuboid* if its ground set can be relabeled $[2n] := \{1, \ldots, 2n\}$ for some integer $n \geq 1$, such that $\{1, 2\}, \{3, 4\}, \ldots, \{2n - 1, 2n\}$ are transversals [8, 10]. In particular, every member has cardinality n. Note that every cuboid without a cover of cardinality one is tangled. Consider, for instance, the clutter

$$Q_6 = \{\{2, 4, 6\}, \{1, 3, 6\}, \{1, 4, 5\}, \{2, 3, 5\}\},$$

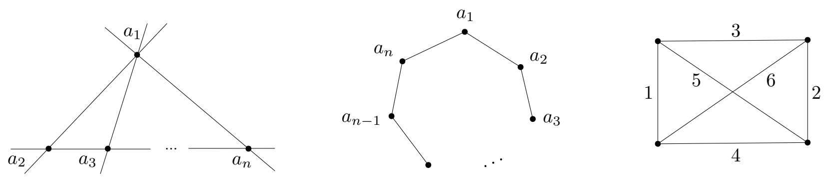

Figure 4.1: **Left:** The members of a *delta* represented the lines of a degenerate projective plane. **Middle:** The minimum cardinality members of an *extended odd hole* represented as the edges of an odd hole. **Right:** The members of Q_6 represented as the triangles of K_4.

whose elements and members correspond to the edges and triangles of the complete graph K_4, as labeled in Figure 4.1. Then Q_6 is a cuboid – as $\{1, 2\}$, $\{3, 4\}$, $\{5, 6\}$ are transversals – without a cover of cardinality one. Moreover, it can be readily checked that Q_6 has no minor that is a delta or the blocker of an extended odd hole. Consequently, Q_6 is a clean tangled clutter.

More generally, clean tangled clutters have been the subject of recent study as they have been instrumental in the progress made towards various outstanding problems on ideal clutters, ranging from recognizing idealness [4], the $\tau = 2$ Conjecture [13] and new examples of ideal minimally non-packing clutters [5], idealness of k-wise intersecting families [11], to the existence of dyadic fractional packings in ideal clutters [9].

Even though their definition is purely combinatorial, clean tangled clutters enjoy fascinating geometric properties, and in this paper we initiate the study of the geometric attributes of such clutters. We prove three results that manifest an interplay between the geometry and the combinatorics of such clutters. In particular, full-dimensional simplices, projective geometries over the two-element field, and an astonishing connection between them play a central role in this work.

The core and the setcore of clean tangled clutters

Let $\mathcal{C}$ be a clutter over ground set V. The *incidence matrix of* $\mathcal{C}$, denoted $M(\mathcal{C})$, is the $0 - 1$ matrix whose columns are indexed by the elements and whose rows are indexed by the members. Consider the primal-dual pair of linear programs

$$(P) \quad \begin{array}{ll} \min & \mathbf{1}^\top x \\ \text{s.t.} & M(\mathcal{C})x \geq \mathbf{1} \\ & x \geq \mathbf{0} \end{array} \qquad (D) \quad \begin{array}{ll} \max & \mathbf{1}^\top y \\ \text{s.t.} & M(\mathcal{C})^\top y \leq \mathbf{0} \\ & y \geq \mathbf{0}. \end{array}$$

The incidence vector of any cover of $\mathcal{C}$ gives a feasible solution for (P). Thus $\tau(\mathcal{C})$ is an upper bound on the optimal value of (P). A *fractional packing of* $\mathcal{C}$ is any feasible solution y for (D), and its *value* is $\mathbf{1}^\top y$. Its *support*, denoted support(y), is the clutter over ground set V whose members are $\{C \in \mathcal{C} : y_C > 0\}$. It follows from Weak LP Duality that every fractional packing has value at most $\tau(\mathcal{C})$. In general, this upper bound is far from being tight. However, what is fascinating about clean clutters is that,

Theorem 4.1.3 ([7], Theorem 3 and [4], Lemma 1.6)**.** *Every clean clutter with covering number at least two has a fractional packing of value two. In particular, every clean tangled clutter has a fractional packing of value two.*

We may therefore make the following definition:

Definition 4.1.4. Let $\mathcal{C}$ be a clean tangled clutter. Then the *core* of $\mathcal{C}$ is the clutter

$$\text{core}(\mathcal{C}) = \{C \in \mathcal{C} : y_C > 0 \text{ for some fractional packing } y \text{ of value two}\}.$$

By Theorem 4.1.3, every clean tangled clutter has a nonempty core. Let us identify the core for two examples of clean tangled clutters. For the first example, consider the clean tangled clutter Q_6. As $\left(\frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}\right) \in \mathbb{R}_+^{Q_6}$ is a fractional packing of value two, it follows that $\text{core}(Q_6) = Q_6$. For the second example, consider the clutter Q whose incidence matrix is

$$M(Q) = \begin{array}{c} \\ \frac{1}{2} \\ \frac{1}{2} \\ \frac{1}{2} \\ \frac{1}{2} \\ 0 \\ 0 \\ 0 \\ 0 \end{array} \begin{array}{c} \begin{array}{cccccccc} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \end{array} \\ \begin{pmatrix} 1 & 1 & 0 & 0 & 1 & 0 & 1 & 0 \\ 1 & 1 & 0 & 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 & 1 & 0 & 0 & 1 \\ 0 & 0 & 1 & 1 & 0 & 1 & 1 & 0 \\ 1 & 0 & 1 & 1 & 1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 & 0 & 1 & 0 & 1 \\ 0 & 1 & 1 & 0 & 1 & 0 & 0 & 1 \\ 1 & 1 & 0 & 1 & 0 & 1 & 1 & 0 \end{pmatrix} \end{array}$$

Q is an *ideal minimally non-packing* clutter with covering number two [13], implying in turn that it is a clean tangled clutter [5]. The row labels indicate the unique fractional packing of value two, where uniqueness is a simple consequence of Complementary Slackness. Subsequently, $\text{core}(Q)$ consists of the four members of Q corresponding to the top four rows of $M(Q)$.

Let $\mathcal{C}$ be a clutter over ground set V. Distinct elements u, v are *duplicates in* $\mathcal{C}$ if the corresponding columns in $M(\mathcal{C})$ are identical. To *duplicate an element* w *of* $\mathcal{C}$ is to introduce a new element $\bar{w}$ and replace $\mathcal{C}$ by the clutter over ground set $V \cup \{\bar{w}\}$ whose members are $\{C : w \notin C \in \mathcal{C}\} \cup \{C \cup \{\bar{w}\} : w \in C \in \mathcal{C}\}$. A *duplication of* $\mathcal{C}$ is any clutter obtained from $\mathcal{C}$ after duplicating some elements.

Looking back at $\text{core}(Q)$, we see that elements $1, 2$ are duplicates and elements $3, 4$ are duplicates, and that $\text{core}(Q)$ is a duplication of Q_6. In fact, the core of any clean tangled clutter is a duplication of a cuboid – let us elaborate.

Take an integer $n \geq 1$ and a set $S \subseteq \{0,1\}^n$. The *cuboid of* S, denoted $\text{cuboid}(S)$, is the clutter over ground set $[2n]$ whose members have incidence vectors $\{(p_1, 1-p_1, \dots, p_n, 1-p_n) : p \in S\}$. In particular, there is a bijection between the members of $\text{cuboid}(S)$ and the points in S. Observe that

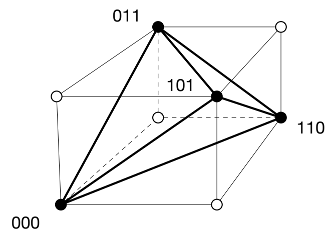

Figure 4.2: A representation of setcore(Q_6) and its convex hull.

every cuboid is obtained in this way. For example, Q_6 is the cuboid of the set $\{000, 110, 101, 011\} \subseteq \{0,1\}^3$, represented in Figure 4.2.

Take a point $q \in \{0,1\}^n$. To *twist S by q* is to replace S by $S \triangle q := \{p \triangle q : p \in S\}$, where the second $\triangle$ denotes coordinate-wise addition modulo 2. Take a coordinate $i \in [n]$. Denote by e_i the i^{th} unit vector of appropriate dimension. To *twist coordinate i of S* is to replace S by $S \triangle e_i$. Two sets S_1, S_2 are *isomorphic*, written as $S_1 \cong S_2$, if one is obtained from the other after relabeling and twisting some coordinates. Two distinct coordinates $i, j \in [n]$ are *duplicates in S* if $S \subseteq \{x : x_i = x_j\}$ or $S \subseteq \{x : x_i + x_j = 1\}$. Observe that if two coordinates are duplicates in a set, then they are duplicates in any isomorphic set. Observe further that S has duplicated coordinates if, and only if, cuboid(S) has duplicated elements.

Let $\mathcal{C}$ be a clean tangled clutter over ground set V. Denote by $G(\mathcal{C})$ the graph over vertex set V whose edges correspond to the minimum covers of $\mathcal{C}$. It can be readily checked that $G(\mathcal{C})$ is bipartite and every vertex of it is incident with an edge [7]. The *rank of $\mathcal{C}$*, denoted rank($\mathcal{C}$), is the number of connected components of the bipartite graph $G(\mathcal{C})$. Our first result, below, justifies this choice of terminology.

For example, the reader can verify that $G(Q_6), G(Q)$ are the bipartite graphs represented in Figure 4.3, each of which has exactly three connected components, so rank$(Q_6) =$ rank$(Q) = 3$.

We are ready to state our first result:

Theorem 4.1.5 (proved in §4.2)**.** *Let $\mathcal{C}$ be a clean tangled clutter of rank r. Then there exists a set $S \subseteq \{0,1\}^r$ such that the following statements hold:*

(i) core($\mathcal{C}$) *is a duplication of* cuboid(S)*, and up to isomorphism, S is the unique set satisfying this property.*

(ii) There is a one-to-one correspondence between the fractional packings of value two in $\mathcal{C}$ and the different ways to express $\frac{1}{2} \cdot \mathbf{1}$ as a convex combination of the points in S.

(iii) conv(S) *is a full-dimensional polytope containing $\frac{1}{2} \cdot \mathbf{1}$ in its interior.*

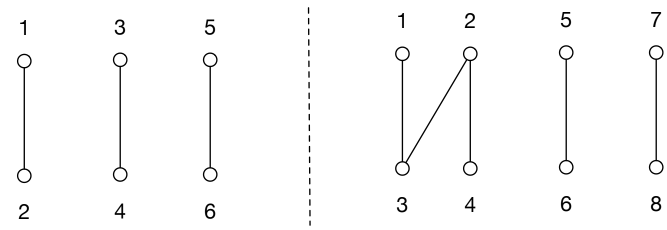

Figure 4.3: **Left:** A representation of $G(Q_6)$. **Right:** A representation of $G(Q)$.

Definition 4.1.6. Let $\mathcal{C}$ be a clean tangled clutter of rank r. The *setcore of* $\mathcal{C}$, denoted setcore($\mathcal{C}$), is the unique set $S \subseteq \{0,1\}^r$ such that core($\mathcal{C}$) is a duplication of cuboid(S).

An explicit description of the setcore is provided in §4.2.

For example, we see that setcore(Q_6) $=$ setcore(Q) $= \{000, 110, 101, 011\}$. Notice further that the convex hull of $\{000, 110, 101, 011\}$ is a full-dimensional polytope containing the point $(\frac{1}{2}, \frac{1}{2}, \frac{1}{2})$ in its interior. (See Figure 4.2.)

Simplices and projective geometries over the two-element field

The convex hull of the setcore of any clean tangled clutter is a full-dimensional polytope by Theorem 4.1.5 (iii). A natural geometric question arises: When is this polytope a simplex? Surprisingly, the answer to this basic question takes us to binary matroids. (Our terminology follows Oxley [20].)

Let $k \geq 1$ be an integer, and let A be the $k \times (2^k - 1)$ matrix whose columns are $\left\{a \in \{0,1\}^k : a \neq \mathbf{0}\right\}$. The binary matroid represented by A is called the ***rank-k projective geometry over*** $GF(2)$ and denoted $PG(k-1,2)$.[1] Let $n := 2^k - 1$. The *cycle space of* $PG(k-1,2)$ is

$$\text{cycle}(PG(k-1,2)) := \left\{x \in \{0,1\}^n : Ax \equiv \mathbf{0} \pmod 2\right\}.$$

Observe that the cycle space forms a vector space over $GF(2)$. The *cocycle space of* $PG(k-1,2)$ is

$$\text{cocycle}(PG(k-1,2)) := \left\{A^\top y \mod 2 : y \in \{0,1\}^k\right\} \subseteq \{0,1\}^n.$$

Observe that the cocycle space is the orthogonal complement of the cycle space. As the rows of A are linearly independent over $GF(2)$, the cocycle space has $2^k = n+1$ points. In fact, we see in §4.3 that the $n+1$ points form the vertices of an n-dimensional simplex. Our second result, below, serves as a converse to this statement.

[1] In the context of binary matroids, rank refers to $GF(2)$-rank, whereas in the context of clutters, rank refers to $\mathbb{R}$-rank.

$$(1) \qquad \begin{pmatrix}1&0&1\\0&1&1\end{pmatrix} \qquad \begin{pmatrix}1&0&0&0&1&1&1\\0&1&0&1&0&1&1\\0&0&1&1&1&0&1\end{pmatrix}$$

Figure 4.4: Representations of $PG(0,2), PG(1,2)$ and $PG(2,2)$, from left to right.

We refer to $PG(k-1,2), k \geq 1$ simply as *projective geometries*. Let us look at the first three projective geometries. For the first one, cocycle$(PG(0,1)) = \{0,1\}$. As for the second one, notice that $PG(1,2)$ is nothing but the graphic matroid of a triangle, and that cocycle$(PG(1,2)) = \{000,110,101,011\}$. The third one, $PG(2,2)$, is known as the *Fano matroid*. See Figure 4.4 for representations of these three matroids.

We are now ready to state our second result:

Theorem 4.1.7 (($\Leftarrow$) proved in §4.3, ($\Rightarrow$) proved in §4.4)**.** *Let $\mathcal{C}$ be a clean tangled clutter. Then* conv(setcore($\mathcal{C}$)) *is a simplex if, and only if,* setcore($\mathcal{C}$) *is the cocycle space of a projective geometry.*

For instance, as can be seen in Figure 4.2, the convex hull of setcore(Q_6) is a simplex, so according to Theorem 4.1.7, setcore(Q_6) is the cocycle space of a projective geometry. This is indeed the case as setcore$(Q_6) = \{000,110,101,011\} =$ cocycle$(PG(1,2))$.

The clutter of the lines of the Fano plane

Consider the clutter over ground set $\{1,\ldots,7\}$ whose members are

$$\mathbb{L}_7 := \big\{\{1,2,3\},\{1,4,5\},\{1,6,7\},\{2,5,6\},\{2,4,7\},\{3,4,6\},\{3,5,7\}\big\}.$$

Note that the members of $\mathbb{L}_7$ correspond to the lines of the *Fano plane*, as displayed in Figure 4.5. The members of $\mathbb{L}_7$ may also be viewed as the lines (i.e. triangles) of the Fano matroid. Observe further that $\mathbb{L}_7$ is an identically self-blocking clutter.

As the only *minimally non-ideal binary clutter* with a member of cardinality three [6], $\mathbb{L}_7$ plays a crucial role in Seymour's *Flowing Conjecture*, predicting an excluded minor characterization of *ideal binary clutters* [23].

Our third result relates to finding $\mathbb{L}_7$ as a minor in clean tangled clutters:

Theorem 4.1.8 (proved in §4.5)**.** *Let $\mathcal{C}$ be a clean tangled clutter where* conv(setcore($\mathcal{C}$)) *is a simplex. If* rank$(\mathcal{C}) > 3$*, then $\mathcal{C}$ has an $\mathbb{L}_7$ minor.*

Let us outline a naive approach for proving this theorem. Though unsuccessful, this attempt explains the intuition behind why Theorem 4.1.8 should be true. Let $\mathcal{C}$ be a clean tangled clutter where conv(setcore($\mathcal{C}$)) is a simplex,

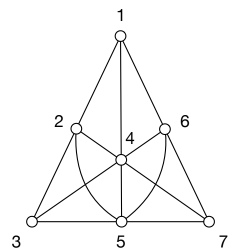

Figure 4.5: The Fano plane

and rank($\mathcal{C}$) > 3. By Theorem 4.1.7, setcore($\mathcal{C}$) $=$ cocycle($PG(k-1,2)$) for some $k \geq 1$. As $2^k - 1 =$ rank($\mathcal{C}$) > 3 and rank($\mathcal{C}$) $= 2^k - 1$, we must have that $k \geq 3$. From here, the reader can verify that since $PG(k-1,2)$ has the Fano matroid as a minor, cuboid(cocycle($PG(k-1,2)$)) has an $\mathbb{L}_7$ minor. Consequently, core($\mathcal{C}$), which is a duplication of cuboid(setcore($\mathcal{C}$)) by Theorem 4.1.5, must have an $\mathbb{L}_7$ minor. However, this does *not* necessarily imply that $\mathcal{C}$ has an $\mathbb{L}_7$ minor, because core($\mathcal{C}$) is only a subset of $\mathcal{C}$, so minors of core($\mathcal{C}$) do not necessarily correspond to minors of $\mathcal{C}$. In §4.5, we see an elaborate, successful attempt for proving Theorem 4.1.8.

Outline of the paper

In §4.2, we prove Theorem 4.1.5 and provide applications of the theorem used in later sections. In §4.3, after a primer on binary matroids, we show how every projective geometry leads to a simplex, and prove Theorem 4.1.7 ($\Leftarrow$) as a consequence. In §4.4, we prove Theorem 4.1.7 ($\Rightarrow$), and derive an appealing consequence on characterizing simplices that come from projective geometries. In §4.5, after laying some ground work, we prove Theorem 4.1.8, and then discuss an application to idealness. Finally, in §4.6, we discuss future directions for research, and conclude with two conjectures.

4.2 The core and the setcore of clean tangled clutters

In this section, after presenting some lemmas, we prove Theorem 4.1.5, and then provide three applications for clean tangled clutters: the first is a characterization of the core, the second is an explicit description of the setcore when the rank is small, and the third is an equivalent condition for having a simplicial setcore.

Given a clean tangled clutter $\mathcal{C}$ over ground set V, recall that $G(\mathcal{C})$ denotes the graph over vertex set V whose edges correspond to the minimum covers of $\mathcal{C}$. We need the following theorem:

Theorem 4.2.1 ([9]). *Let $\mathcal{C}$ be a clean tangled clutter. Then $G(\mathcal{C})$ is a bipartite graph where every vertex is incident with an edge. Moreover, if $G(\mathcal{C})$ is* *not connected and* $\{U, U'\}$ *is the bipartition of a connected component, then* $\mathcal{C} \setminus U/U'$ *is a clean tangled clutter.*

Some lemmas

Let $\mathcal{C}$ be a clutter over ground set V. Consider the primal-dual pair of linear programs

$$\begin{array}{llll} & \min & \mathbf{1}^\top x & \\ (P) & \text{s.t.} & \sum_{v\in C} x_v \geq 1 & C \in \mathcal{C} \\ & & x \geq \mathbf{0} & \\ & \max & \mathbf{1}^\top y & \\ (D) & \text{s.t.} & \sum (y_C : v \in C \in \mathcal{C}) \leq 1 & v \in V \\ & & y \geq \mathbf{0}. & \end{array}$$

By applying Complementary Slackness to this pair, we get the following:

Remark 4.2.2. *Let* $\mathcal{C}$ *be a clutter,* B *a minimum cover, and* y *a fractional packing of value* $\tau(\mathcal{C})$*, if there is any. Then* $|C \cap B| = 1$ *for every* $C \in \mathcal{C}$ *such that* $y_C > 0$*, and* $\sum (y_C : v \in C \in \mathcal{C}) = 1$ *for every element* $v \in B$*.*

An explicit description of the setcore. Let $\mathcal{C}$ be a clean tangled clutter. Recall that

$$\text{core}(\mathcal{C}) = \{C \in \mathcal{C} : y_C > 0 \text{ for some fractional packing } y \text{ of value two}\}.$$

The following is an immediate consequence of Remark 4.2.2:

Remark 4.2.3. *Let* $\mathcal{C}$ *be a clean tangled clutter over ground set* V*. Then every member of* core($\mathcal{C}$) *is a transversal of the minimum covers of* $\mathcal{C}$*. Moreover, for every fractional packing* y *of value two and for every element* $v \in V$*,* $\sum (y_C : v \in C \in \mathcal{C}) = 1$.

Let $G := G(\mathcal{C})$ and $r := \text{rank}(\mathcal{C})$. By Theorem 4.2.1, G is a bipartite graph where every vertex is incident with an edge, and it has r connected components by definition. For each $i \in [r]$, denote by $\{U_i, V_i\}$ the bipartition of the i^{th} connected component of G. As an immediate consequence of Remark 4.2.3,

Remark 4.2.4. *Let* $\mathcal{C}$ *be a clean tangled clutter of rank* r*, and for each* $i \in [r]$*, denote by* $\{U_i, V_i\}$ *the bipartition of the* i^{th} *connected component of* $G(\mathcal{C})$*. Let* C *be a member of* $\mathcal{C}$*. If* $C \in \text{core}(\mathcal{C})$*, then* $C \cap (U_i \cup V_i) \in \{U_i, V_i\}$ *for each* $i \in [r]$*.* [2]

In particular, core($\mathcal{C}$) is a duplication of a cuboid – let us elaborate. Consider the set $S \subseteq \{0,1\}^r$ defined as follows: start with $S = \emptyset$, and for each $C \in \text{core}(\mathcal{C})$, add a point p to S such that

$$p_i = 0 \quad \Leftrightarrow \quad C \cap (U_i \cup V_i) = U_i \quad \forall i \in [r].$$

[2] By the end of this section, we shall see that the converse of this remark also holds.

By Remark 4.2.4, the set S is well-defined and core($\mathcal{C}$) is a duplication of cuboid(S). Thus this must be the unique set foreseen by Theorem 4.1.5 (i), and called the setcore of $\mathcal{C}$ by Definition 4.1.6. We call S the *setcore of $\mathcal{C}$ with respect to* $(U_1, V_1; U_2, V_2; \ldots; U_r, V_r)$, and denote it

$$\text{setcore}(\mathcal{C} : U_1, V_1; U_2, V_2; \ldots; U_r, V_r) .$$

(Note that we have not yet proved uniqueness, though we will see a proof shortly.)

Fractional packings vs. convex combinations. The following remark, which is an immediate consequence of Remark 4.2.2, sheds light on how the hypercube center point $\frac{1}{2} \cdot \mathbf{1}$ comes into play in Theorem 4.1.5:

Remark 4.2.5. *Take an integer $r \geq 1$, a set $S \subseteq \{0,1\}^r$ and let $\mathcal{C} :=$ cuboid(S). Let $y \in \mathbb{R}_+^{\mathcal{C}}$ and define $\alpha \in \mathbb{R}_+^S$ as follows: for every point $p \in S$ and corresponding member $C \in \mathcal{C}$, let $\alpha_p := \frac{1}{2} \cdot y_C$. Then y is a fractional packing of $\mathcal{C}$ of value two if, and only if, $\mathbf{1}^\top \alpha = 1$ and $\sum_{p \in S} \alpha_p \cdot p = \frac{1}{2} \cdot \mathbf{1}$. In particular, $\mathcal{C}$ has a fractional packing of value two if, and only if, $\frac{1}{2} \cdot \mathbf{1} \in$ conv(S).*

Recursive construction of fractional packings. Let $\mathcal{C}$ be a clean tangled clutter where $G(\mathcal{C})$ is not connected, and let $\{U, U'\}$ be the bipartition of a connected component of $G(\mathcal{C})$. Let $\mathcal{C}' := \mathcal{C} \setminus U/U'$. Observe that every member of $\mathcal{C}$ disjoint from U contains U', implying in turn that $\mathcal{C}' = \{C - U' : C \in \mathcal{C}, C \cap U = \emptyset\}$. Observe further that $\mathcal{C}'$ is a clean tangled clutter by Theorem 4.2.1, so it has a fractional packing of value two by Theorem 4.1.3. These observations are used to set up the following lemma:

Lemma 4.2.6. *Let $\mathcal{C}$ be a clean tangled clutter, where $G(\mathcal{C})$ is not connected. Let $\{U, U'\}$ be the bipartition of a connected component of $G(\mathcal{C})$, and let z, z' be fractional packings of $\mathcal{C} \setminus U/U', \mathcal{C}/U \setminus U'$ of value two, respectively. Let $y, y' \in \mathbb{R}_+^{\mathcal{C}}$ be defined as follows:*

$$y_C := \begin{cases} z_{C-U'} & \text{if } C \cap U = \emptyset \\ 0 & \text{otherwise} \end{cases} \quad \text{and} \quad y'_C := \begin{cases} z'_{C-U} & \text{if } C \cap U' = \emptyset \\ 0 & \text{otherwise.} \end{cases}$$

Then $\frac{1}{2}y + \frac{1}{2}y'$ is a fractional packing of $\mathcal{C}$ of value two. In particular,

$$\text{core}(\mathcal{C} \setminus U/U') \subseteq \text{core}(\mathcal{C}) \setminus U/U'.$$

Proof. We leave this as an exercise for the reader. □

Duplicated elements of the core. For the next lemma, we need the following remark:

Remark 4.2.7. *The core of any clean tangled clutter has covering number two.*

Lemma 4.2.8. *Let $\mathcal{C}$ be a clean tangled clutter over ground set V. Then two elements $u, v \in V$ are duplicates in* core$(\mathcal{C})$ *if, and only if, u, v belong to the same part of the bipartition of a connected component of $G(\mathcal{C})$.*

Proof. ($\Leftarrow$) follows Remark 4.2.4. ($\Rightarrow$) By Remark 4.2.4, it suffices to show that u, v belong to the same connected component of $G := G(\mathcal{C})$. Suppose otherwise. In particular, G is not connected. Let $\{U, U'\}$ be the bipartition of the connected component containing u where $u \in U'$. Then $\mathcal{C} \setminus U/U'$ is a clean tangled clutter by Theorem 4.2.1. Let w be a neighbour of u in G; so $w \in U$. Then $\{w, u\}$ is a cover of $\mathcal{C}$. As every member of core$(\mathcal{C})$ containing u also contains v, it follows that $\{w, v\}$ is a cover of core$(\mathcal{C})$, implying in turn that core$(\mathcal{C})\setminus U/U'$ has $\{v\}$ as a cover. However, core$(\mathcal{C} \setminus U/U') \subseteq$ core$(\mathcal{C})\setminus U/U'$ by Lemma 4.2.6, so core$(\mathcal{C} \setminus U/U')$ has a cover of cardinality one, a contradiction to Remark 4.2.7. □

Proof of Theorem 4.1.5

Let $\mathcal{C}$ be a clean tangled clutter over ground set V, let $G := G(\mathcal{C})$, and let $r :=$ rank$(\mathcal{C})$. Recall that r is the number of connected components of G. For each $i \in [r]$, let $\{U_i, V_i\}$ be the bipartition of the i^{th} connected component of G. Let $S :=$ setcore$(\mathcal{C} : U_1, V_1; U_2, V_2; \ldots; U_r, V_r)$. We claim that S satisfies (i)-(iii) of Theorem 4.1.5, thereby finishing the proof. It follows from the construction of S that core$(\mathcal{C})$ is a duplication of cuboid(S). Moreover, by Lemma 4.2.8, S is the unique set satisfying this property, up to isomorphism. Thus, **(i)** holds. **(ii)** Every fractional packing of value two in $\mathcal{C}$ corresponds to a fractional packing of value two in core$(\mathcal{C})$, and every fractional packing of value two in core$(\mathcal{C})$ corresponds to a fractional packing of value two in cuboid(S). By Remark 4.2.5, the fractional packings of value two in cuboid(S) are in correspondence with the different ways to express $\frac{1}{2} \cdot \mathbf{1}$ as a convex combination of the points in S. These observations prove (ii).

Claim 1. $\frac{1}{2} \cdot \mathbf{1} \in$ conv(S).

Proof of Claim. By Theorem 4.1.3, $\mathcal{C}$ and therefore core$(\mathcal{C})$ has a fractional packing of value two, implying that cuboid(S) has a fractional packing of value two. It therefore follows from Remark 4.2.5 that $\frac{1}{2} \cdot \mathbf{1}$ can be expressed as a convex combination of the points in S, thereby proving the claim. ◊

Claim 2. $\frac{1}{2} \cdot \mathbf{1} \pm \frac{1}{2} \cdot e_i \in$ conv(S) *for each* $i \in [r]$.

Proof of Claim. When $r = 1$, note that Claim 1 implies that $S = \{0, 1\}$, so Claim 2 holds. Now assume $r \geq 2$. Let $\mathcal{C}' := \mathcal{C} \setminus U_i/V_i$. Then $\mathcal{C}'$ is a clean tangled clutter by Theorem 4.2.1, and core$(\mathcal{C}') \subseteq$ core$(\mathcal{C}) \setminus U_i/V_i$ by Lemma 4.2.6. Let z be a fractional packing of $\mathcal{C}'$ of value two. Then by Remark 4.2.3,

$$\sum (z_{C'} : v \in C' \in \mathcal{C}') = 1 \quad \forall\, v \in V - (U_i \cup V_i).$$

Define $y \in \mathbb{R}_+^{\mathcal{C}}$ as follows:

$$y_C := \begin{cases} z_{C-V_i} & \text{if } C \cap U_i = \emptyset \\ 0 & \text{otherwise.} \end{cases}$$

Notice that

$$\begin{aligned} \mathbf{1}^\top y &= 2 \\ \sum (y_C : v \in C \in \mathcal{C}) &= 1 \quad \forall\, v \in V - (U_i \cup V_i) \\ \sum (y_C : v \in C \in \mathcal{C}) &= 2 \quad \forall\, v \in V_i \\ \sum (y_C : v \in C \in \mathcal{C}) &= 0 \quad \forall\, v \in U_i. \end{aligned}$$

As support$(z) \subseteq$ core$(\mathcal{C}') \subseteq$ core$(\mathcal{C}) \backslash U_i / V_i$, it follows that support$(y) \subseteq$ core$(\mathcal{C})$. Define $\alpha \in \mathbb{R}_+^S$ as follows: for every point $p \in S$ and corresponding member $C \in$ core$(\mathcal{C})$, let $\alpha_p := \frac{1}{2} \cdot y_C$. Then the equalities above show that $\mathbf{1}^\top \alpha = 1$ and $\sum_{p \in S} \alpha_p \cdot p = \frac{1}{2} \cdot \mathbf{1} + \frac{1}{2} \cdot e_i$. In particular, $\frac{1}{2} \cdot \mathbf{1} + \frac{1}{2} \cdot e_i \in$ conv(S). Repeating the argument on $\mathcal{C}/U_i \setminus V_i$ yields $\frac{1}{2} \cdot \mathbf{1} - \frac{1}{2} \cdot e_i \in$ conv(S), thereby proving the claim. ◊

Claims 1 and 2 together imply that conv(S) is a full-dimensional polytope containing $\frac{1}{2} \cdot \mathbf{1}$ in its interior, so **(iii)** holds. This finishes the proof of Theorem 4.1.5. □

Applications

As the first application of Theorem 4.1.5, we give the following characterization of the core of a clean tangled clutter. Note that this result is the converse of Remark 4.2.4.

Theorem 4.2.9. *Let $\mathcal{C}$ be a clean tangled clutter of rank r, and for each $i \in [r]$, denote by $\{U_i, V_i\}$ the bipartition of the i^{th} connected component of $G(\mathcal{C})$. Then*

$$\text{core}(\mathcal{C}) = \{C \in \mathcal{C} : C \cap (U_i \cup V_i) = U_i \textit{ or } V_i \quad \forall\, i \in [r]\}.$$

Proof. Denote by $\mathcal{C}'$ the clutter on the right-hand side. Let

$$S := \text{setcore}(\mathcal{C} : U_1, V_1; \ldots; U_r, V_r)\,.$$

Let S' be the subset of $\{0,1\}^r$ defined as follows: start with $S' = \emptyset$, and for each $C \in \mathcal{C}'$, add a point p to S' such that

$$p_i = 0 \quad \Leftrightarrow \quad C \cap (U_i \cup V_i) = U_i \quad \forall\, i \in [r].$$

By Remark 4.2.4,

$$\text{core}(\mathcal{C}) \subseteq \mathcal{C}'$$

so $S \subseteq S'$. We know from Theorem 4.1.5 (iii) that $\frac{1}{2} \cdot \mathbf{1}$ lies in the interior of conv(S), so $\frac{1}{2} \cdot \mathbf{1}$ lies in the interior of conv(S'). As a result, for every $p \in S'$,

$\frac{1}{2} \cdot \mathbf{1}$ can be written as a convex combination of the points in S' such that the coefficient of p is nonzero. That is, by Remark 4.2.5, for each $C \in \mathcal{C}'$, there is a fractional packing of $\mathcal{C}'$ whose support includes C. As every fractional packing of $\mathcal{C}'$ is also a fractional packing of $\mathcal{C}$, it follows that

$$\mathcal{C}' \subseteq \text{core}(\mathcal{C})$$

thereby finishing the proof of Theorem 4.2.9. □

For the next application, we give an explicit description of the setcore when the rank is small:

Theorem 4.2.10. *Let $\mathcal{C}$ be a clean tangled clutter with rank r. For each $i \in [r]$, denote by $\{U_i, V_i\}$ the bipartition of the i^{th} connected component of $G(\mathcal{C})$. Then the following statements hold:*

(i) *If $r = 1$, then* $\text{setcore}(\mathcal{C}) = \{0, 1\}$, *and so* $\text{core}(\mathcal{C}) = \{U_1, V_1\}$.

(ii) *If $r = 2$, then* $\text{setcore}(\mathcal{C}) = \{00, 10, 01, 11\}$, *and so* $\text{core}(\mathcal{C}) = \{U_1 \cup U_2, U_1 \cup V_2, V_1 \cup U_2, V_1 \cup V_2\}$.

(iii) *If $r = 3$ and $\mathcal{C}$ does not have disjoint members, then*

$$\text{setcore}(\mathcal{C}) = \{000, 110, 101, 011\} \textit{ or } \{100, 010, 001, 111\}$$

and so

$$\begin{aligned}\text{core}(\mathcal{C}) = &\{U_1 \cup U_2 \cup U_3, U_1 \cup V_2 \cup V_3, V_1 \cup U_2 \cup V_3, V_1 \cup V_2 \cup U_3\} \\ &\textit{or } \{U_1 \cup U_2 \cup V_3, U_1 \cup V_2 \cup U_3, V_1 \cup U_2 \cup U_3, V_1 \cup V_2 \cup V_3\}\end{aligned}$$

Proof. Let $S := \text{setcore}(\mathcal{C}) \subseteq \{0, 1\}^r$. By Theorem 4.1.5 (iii), $\frac{1}{2} \cdot \mathbf{1}$ lies in the interior of $\text{conv}(S)$. This immediately **(i)** and **(ii)**. If $\mathcal{C}$ does not have disjoint members, then neither does $\text{core}(\mathcal{C})$, implying that S does not have antipodal points. These two facts imply **(iii)**. □

Finally, Theorem 4.1.5 allows us to restate the assumption that the setcore of a clean tangled clutter has a simplicial convex hull. This restatement is crucial for the proof of Theorem 4.1.7.

Theorem 4.2.11. *Let $\mathcal{C}$ be a clean tangled clutter. Then* $\text{conv}(\text{setcore}(\mathcal{C}))$ *is a simplex if, and only if, $\mathcal{C}$ has a unique fractional packing of value two.*

Proof. Let $S := \text{setcore}(\mathcal{C})$. Theorem 4.1.5 (ii) states that there is a one-to-one correspondence between the fractional packings of value two in $\mathcal{C}$ and the different ways to describe $\frac{1}{2} \cdot \mathbf{1}$ as a convex combination of the points in S. As a consequence, $\mathcal{C}$ has a unique fractional packing of value two if, and only if, $\frac{1}{2} \cdot \mathbf{1}$ can be written as a unique combination of the points in S. Since $\frac{1}{2} \cdot \mathbf{1}$ lies in the interior of $\text{conv}(S)$ by Theorem 4.1.5 (iii), the theorem follows. □

4.3 From projective geometries to simplices

In this section, we show that the cocycle space of every projective geometry forms a simplex, and then prove Theorem 4.1.7 ($\Leftarrow$) as an immediate consequence.

A primer on binary spaces and binary matroids

Take an integer $n \geq 1$ and a set $S \subseteq \{0,1\}^n$. We say that S is an *affine vector space over* $GF(2)$, or simply an *affine binary space*, if $a \triangle b \triangle c \in S$ for all points $a, b, c \in S$. If S contains $\mathbf{0}$, then S is called a *binary space*. Binary spaces enjoy the following transitive property:

Remark 4.3.1. $S = S \triangle a$ *for every binary space* S *and every point* $a \in S$.

Suppose S is a binary space. By Basic Linear Algebra, there is a $0-1$ matrix A with n columns such that $S = \{x : Ax \equiv \mathbf{0} \pmod 2\}$. Let M be the binary matroid over ground set $EM := [n]$ that is represented by A. The *cycle space of* M is the set cycle$(M) := S$ and the *cocycle space of* M, denoted cocycle$(M) \subseteq \{0,1\}^n$, is the row space of A over $GF(2)$. Notice that cycle(M) , cocycle(M) are binary spaces that are orthogonal complements. Observe that the binary matroid M can be fully determined by either A, its cycle space or its cocycle space.

A *cycle of* M is a subset $C \subseteq EM$ such that $\chi_C \in$ cycle(M), and a *cocycle of* M is a subset $D \subseteq EM$ such that $\chi_D \in$ cocycle(M). In particular, $\emptyset$ is both a cycle and a cocycle. Notice that every cycle and every cocycle have an even number of elements in common. A *circuit of* M is a nonempty cycle that does not contain another nonempty cycle, and a *cocircuit of* M is a nonempty cocycle that does not contain another nonempty cocycle. It is well-known that every cycle is either empty or the disjoint union of some circuits, and that every cocycle is either empty or the disjoint union of some cocircuits [20]. Observe that the cycles, circuits, cocycles, and cocircuits of M correspond respectively to the cocycles, cocircuits, cycles, and circuits of the dual matroid $M^\star$.

An element $e \in EM$ is a *loop of* M if $\{e\}$ is a circuit, and it is a *coloop of* M if $\{e\}$ is a cocircuit. Two distinct elements $e, f \in EM$ are *parallel in* M if $\{e, f\}$ is a circuit. M is a *simple* binary matroid if it has no loops and no parallel elements, i.e. if every circuit has cardinality at least three. A *triangle in* M is a circuit of cardinality three.

Remark 4.3.2. *Take an integer* $n \geq 1$ *and a binary space* $S \subseteq \{0,1\}^n$*, and let* M *be the binary matroid whose cycle space is* S*. Then the points in* S *do not agree on a coordinate if, and only if,* M *has no coloops. Moreover, if the points in* S *do not agree on a coordinate, then* $|S \cap \{x : x_i = 0\}| = |S \cap \{x : x_i = 1\}|$ *for all* $i \in [n]$.

Proof of Theorem 4.1.7 ($\Leftarrow$)

Take an integer $k \geq 1$, and let A be the $k \times (2^k - 1)$ matrix whose columns are all the distinct $0-1$ vectors of dimension k that are nonzero. Recall that $PG(k-1,2)$ is the binary matroid represented by A, cycle$(PG(k-1,2)) = \{x : Ax \equiv \mathbf{0} \pmod 2\}$ and cocycle$(PG(k-1,2))$ is the row space of A generated over $GF(2)$. Recall further that $|\text{cocycle}(PG(k-1,2))| = 2^k$. As A has no zero column, and no two columns of it are equal, $PG(k-1,2)$ is a simple binary matroid. In particular, the points in cocycle$(PG(k-1,2))$ do not agree on a coordinate by Remark 4.3.2.

Proposition 4.3.3. *Take an integer $k \geq 2$. Then the following statements hold for $PG(k-1,2)$:*

(i) every nonempty cocycle has cardinality 2^{k-1},

(ii) every two elements appear together in a triangle,

(iii) every cycle is the symmetric difference of some triangles.

Proof. **(i)** Let D be a nonempty cocycle. Then χ_D is nonzero and belongs to cocycle$(PG(k-1,2))$. Let A' be a $k \times (2^k - 1)$ matrix with $0-1$ entries whose first row is χ_D and whose rows form a basis for cocycle$(PG(k-1,2))$ over $GF(2)$. Notice that the orthogonal complement of cocycle$(PG(k-1,2))$ over $GF(2)$ is equal to

$$\text{cycle}(PG(k-1,2)) = \{x : A'x \equiv \mathbf{0} \pmod 2\}.$$

As $PG(k-1,2)$ is a simple binary matroid, it follows that A' has no zero column, and no two columns of it are equal. As A' has $2^k - 1$ columns and k rows, it follows that the columns of A' are all the $0-1$ vectors of dimension k that are nonzero. In particular, every row of A' has 2^{k-1} ones and $2^{k-1} - 1$ zeros. In particular, $|D| = 2^{k-1}$.

(ii) Let A be the $k \times (2^k - 1)$ matrix representation of $PG(k-1,2)$ whose columns are all the $0-1$ vectors of dimension k that are nonzero. Pick distinct elements e, f of $PG(k-1,2)$, and let a, b be the corresponding columns of A. Notice that $a+b \pmod 2$ is another column of A, and let g be the corresponding element of $PG(k-1,2)$. Then $\{e,f,g\}$ is the desired triangle of $PG(k-1,2)$.

(iii) Let C be a nonempty cycle. We proceed by induction on $|C| \geq 3$. The base $|C| = 3$ holds trivially. For the induction step assume that $|C| \geq 4$. Pick distinct elements $e, f \in C$. By (ii) there is an element g such that $\{e,f,g\}$ is a triangle. Since $C \triangle \{e,f,g\}$ is a cycle of smaller cardinality than C, the induction hypothesis applies and tell us that $C \triangle \{e,f,g\}$ is the symmetric difference of some triangles, implying in turn that C is the symmetric difference of some triangles, thereby completing the induction step. $\square$

We are now ready to present the key result of this subsection:

Theorem 4.3.4. *Take an integer* $k \geq 1$ *and let* $S := \text{cocycle}(PG(k-1, 2))$*. Then* $\text{conv}(S)$ *is a full-dimensional simplex containing* $\frac{1}{2} \cdot \mathbf{1}$ *in its interior. In particular,* $\frac{1}{2^{k-1}} \cdot \mathbf{1}$ *is the unique fractional packing of* $\text{cuboid}(S)$ *of value two.*

Proof. Let $n := 2^k - 1$. We know that S is a subset of $\{0, 1\}^n$ and has exactly $n + 1$ points. It follows from Proposition 4.3.3 (i) that the inequality

$$\sum_{i=1}^{n} x_i \leq \frac{n+1}{2}$$

is valid for $\text{conv}(S)$, and that every point in S except for $\mathbf{0}$ satisfies this inequality at equality. As S is a binary space, $S \triangle p = S$ for every point $p \in S$ by Remark 4.3.1. This transitive property implies that for each $p \in S$, the transformed inequality

$$\sum_{i:p_i=0} x_i + \sum_{j:p_j=1} (1 - x_j) \leq \frac{n+1}{2}$$

is also valid for $\text{conv}(S)$, and every point in S except for p satisfies this inequality at equality. Hence, $\text{conv}(S)$ is an n-dimensional simplex whose $n + 1$ facets are as described above.

As the point $\frac{1}{2} \cdot \mathbf{1}$ satisfies every inequality strictly, it lies in the interior of $\text{conv}(S)$. In fact, as S is a binary space whose points do not agree on a coordinate, $|S \cap \{x : x_i = 0\}| = |S \cap \{x : x_i = 1\}|$ for each $i \in [n]$ by Remark 4.3.2, so

$$\sum_{p \in S} \frac{1}{n+1} \cdot p = \frac{1}{2} \cdot \mathbf{1}.$$

As $\text{conv}(S)$ is a simplex, it follows from Remark 4.2.5 that $\frac{2}{n+1} \cdot \mathbf{1} = \frac{1}{2^{k-1}} \cdot \mathbf{1}$ is the unique fractional packing of $\text{cuboid}(S)$ of value two, thereby finishing the proof of Theorem 4.3.4. □

As a consequence,

Proof of Theorem 4.1.7 ($\Leftarrow$). Let $\mathcal{C}$ be a clean tangled clutter. If $\text{setcore}(\mathcal{C})$ is the cocycle space of a projective geometry, then Theorem 4.3.4 implies that $\text{conv}(\text{setcore}(\mathcal{C}))$ is a simplex, as required. □

4.4 From simplices to projective geometries

In this section, after presenting a lemma on constructing projective geometries, we prove Theorem 4.1.7 ($\Rightarrow$), and then present an appealing consequence characterizing when a simplex comes from a projective geometry.

We start with the following key lemma allowing for an inductive argument:

Lemma 4.4.1. *Let $\mathcal{C}$ be a clean tangled clutter with a unique fractional packing of value two. Suppose $G(\mathcal{C})$ is not connected, and let $\{U, U'\}$ be the bipartition of some connected component of $G(\mathcal{C})$. Then $\mathcal{C} \setminus U/U'$ is a clean tangled clutter with a unique fractional packing of value two. Moreover, if y, z are the fractional packings of $\mathcal{C}, \mathcal{C} \setminus U/U'$ of value two, respectively, then* $\text{support}(z) = \text{support}(y) \setminus U/U'$.

Proof. By Theorem 4.2.1, $\mathcal{C} \setminus U/U'$ and $\mathcal{C}/U \setminus U'$ are clean tangled clutters, so we may apply Theorem 4.1.3 and conclude that they have fractional packings z, z' of value two, respectively. Let $t, t' \in \mathbb{R}_+^{\mathcal{C}}$ be defined as follows:

$$t_C := \begin{cases} z_{C-U'} & \text{if } C \cap U = \emptyset \\ 0 & \text{otherwise} \end{cases} \quad \text{and} \quad t'_C := \begin{cases} z'_{C-U} & \text{if } C \cap U' = \emptyset \\ 0 & \text{otherwise.} \end{cases}$$

By Lemma 4.2.6, $\frac{1}{2}t + \frac{1}{2}t'$ is a fractional packing of $\mathcal{C}$ of value two. It therefore follows from the uniqueness assumption that $\frac{1}{2}t + \frac{1}{2}t' = y$. Subsequently, z must be the unique fractional packing of $\mathcal{C} \setminus U/U'$ of value two, z' must be the unique fractional packing of $\mathcal{C}/U \setminus U'$ of value two, and

$$\begin{aligned} \text{support}(z) &= \text{support}(y) \setminus U/U' \\ \text{support}(z') &= \text{support}(y)/U \setminus U', \end{aligned}$$

as desired. □

Constructing projective geometries

For an integer $r \geq 1$ and a set $S \subseteq \{0,1\}^r$, the *incidence matrix of S* is the matrix whose rows are the points in S. Denote by J the all-ones matrix of appropriate dimensions. Take an integer $k \geq 1$ and let A be the incidence matrix of cocycle$(PG(k-1,2))$. Then every column of A has 2^{k-1} ones and 2^{k-1} zeros. In fact, A has the following recursive description:

Remark 4.4.2. *Take an integer $k \geq 2$. If A' is the incidence matrix of* cocycle$(PG(k-2,2))$*, then up to permuting rows and columns,*

$$\begin{pmatrix} \mathbf{1} & A' & J - A' \\ \mathbf{0} & A' & A' \end{pmatrix}$$

is the incidence matrix of cocycle$(PG(k-1,2))$*. Moreover, every element of $PG(k-1,2)$ can be used as the left-most column in the incidence matrix above.*

Consequently, for every pair a, b of columns of A,

$$\begin{aligned} |\{j : a_j = b_j = 0\}| &= |\{j : a_j = b_j = 1\}| \\ &= |\{j : a_j = 1, b_j = 0\}| \\ &= |\{j : a_j = 0, b_j = 1\}| \\ &= 2^{k-2}. \end{aligned}$$

Two columns of a $0-1$ matrix are *complementary* if they add up to the all-ones vector. If $\mathcal{C}$ is the cuboid of cocycle$(PG(k-1,2))$, then every column of $M(\mathcal{C})$ has 2^{k-1} ones, and by the expressions above, every pair of columns of $M(\mathcal{C})$ are either complementary or have exactly 2^{k-2} ones in common.

Remark 4.4.3. *Take an integer $k \geq 2$, and let*

$$\mathcal{C} := \text{cuboid}(\text{cocycle}(PG(k-1,2))).$$

Then for every minimum cover $\{u,v\}$ of $\mathcal{C}$, the minor $\mathcal{C} \setminus u/v$ is obtained from cuboid(cocycle$(PG(k-2,2))$) *after duplicating every element once.*

This remark is an immediate consequence of Remark 4.4.2, and is helpful to keep in mind when parsing the hypotheses of the following lemma, which is the main result of this section:

Lemma 4.4.4. *Take an integer $r \geq 2$ and a clutter $\mathcal{C}$ whose ground set V is partitioned into nonempty parts $U_1, V_1, \ldots, U_r, V_r$ such that*

- *the elements in each part are duplicates,*
- *for each $i \in [r]$, if $u \in U_i$ and $v \in V_i$, then $\{u,v\}$ is a transversal of $\mathcal{C}$, and*
- *for each $i \in [r]$, $\mathcal{C} \setminus U_i/V_i$ (resp. $\mathcal{C}/U_i \setminus V_i$) is a duplication of the cuboid of the cocycle space of a projective geometry.*

Assume further that $\mathcal{C}$ has exactly $r+1$ members and a unique fractional packing of value two. Then there is an integer $k \geq 2$ such that $r = 2^k - 1$ and $\mathcal{C}$ is a duplication of cuboid(cocycle$(PG(k-1,2))$).

Proof. We may assume after contracting some duplicate elements that $U_i = \{u_i\}$ and $V_i = \{v_i\}$ for each $i \in [r]$. In particular, $\mathcal{C}$ is a cuboid. As $\mathcal{C}$ has a fractional packing of value two, it follows that $\tau(\mathcal{C}) \geq 2$, so $\mathcal{C}$ is a tangled clutter. For each $i \in [r]$, let $f(u_i) := v_i$ and $f(v_i) := u_i$.

Claim 1. *$\mathcal{C}$ does not have duplicated elements. In particular, if $\{u,v\}$ is a transversal of $\mathcal{C}$, then $v = f(u)$.*

Proof of Claim. Suppose for a contradiction that u, u' are duplicates. Since $\tau(\mathcal{C}) = 2$, $\{u,u'\}$ is not a cover, so $u' \neq f(u)$. But then $\mathcal{C} \setminus f(u)/u$ has $\{u'\}$ as a cover, a contradiction as $\mathcal{C} \setminus f(u)/u$ is a duplication of the cuboid of the cocycle space of a projective geometry. ♢

In what follows the reader should keep in mind that our labeling of the columns of $M(\mathcal{C})$ induces a labeling for the columns of $M(\mathcal{C} \setminus f(u)/u)$, for each $u \in V$. In particular, $M(\mathcal{C} \setminus f(u)/u)$ and $M(\mathcal{C}/f(u) \setminus u)$ have the same column labels, for each $u \in V$.

Claim 2. *There is an integer $k \geq 2$ such that the following statements hold:*

(1) for each $u \in V$, $\mathcal{C} \setminus f(u)/u$ is a duplication of cuboid(cocycle$(PG(k-2,2))$),

(2) $|\mathcal{C}| = 2^k$*,*

(3) every column of $M(\mathcal{C})$ *has exactly* 2^{k-1} *ones,*

(4) every pair of columns of $M(\mathcal{C})$ *are either complementary or have exactly* 2^{k-2} *ones in common.*

Proof of Claim. For each $u \in V$, $\mathcal{C} \setminus f(u)/u$ is a duplication of

$$\text{cuboid}(\text{cocycle}(PG(k_u - 2, 2)))$$

for some integer $k_u \geq 2$. In particular, every column u of $M(\mathcal{C})$ has exactly $2^{k_u-1} = |\text{cocycle}(PG(k_u - 2, 2))\,|$ ones. Notice now that if $u \in V$ and $w \in V - \{u, f(u)\}$, then the number of ones in column w of $M(\mathcal{C})$ is equal to the sum of the number of ones in column w of $M(\mathcal{C} \setminus f(u)/u)$ and the number of ones in column w of $M(\mathcal{C}/f(u) \setminus u)$, so

$$2^{k_w-1} = 2^{k_u-2} + 2^{k_{f(u)}-2}$$

implying in turn that $k_w = k_u = k_{f(u)}$. As a result, $(k_u : u \in V)$ are all equal to k for some integer $k \geq 2$. It can be readily checked that (1)-(4) hold for k, as required. ◊

Following up on Claim 2 (4), if a pair of columns of $M(\mathcal{C})$ are complementary, then by Claim 1, the column labels must be $u, f(u)$ for some $u \in V$.

Claim 3. $\frac{1}{2^{k-1}} \cdot \mathbf{1} \in \mathbb{R}^{\mathcal{C}}$ *is the unique fractional packing of* $\mathcal{C}$ *of value two.*

Proof of Claim. This follows from Claim 2 (2)-(3) and our assumption that $\mathcal{C}$ has a unique fractional packing of value two. ◊

Claim 4. *The following statements hold for every* $u \in V$*:*

(1) a pair of identical columns in the matrix $M(\mathcal{C} \setminus f(u)/u)$ *correspond to a complementary pair of columns in the matrix* $M(\mathcal{C}/f(u) \setminus u)$*,*

(2) $M(\mathcal{C} \setminus f(u)/u)$ *does not have three identical columns,*

(3) $r = 2^k - 1$*, and*

(4) $\mathcal{C} \setminus f(u)/u$ *is obtained from* cuboid(cocycle($PG(k-2, 2)$)) *after duplicating every element exactly once.*

Proof of Claim. **(1)** follows from Claim 2 (4). **(2)** follows from (1). **(3)** follows from Claim 2 (2) and our assumption that $|\mathcal{C}| = r + 1$. **(4)** Claim 2 (1), together with part (2) of this claim, implies that the minor $\mathcal{C} \setminus f(u)/u$ is obtained from cuboid(cocycle($PG(k-2, 2)$)) after duplicating every element at most once. In particular,

$$2r = |V| \leq 2 + 2 \cdot 2 \cdot (2^{k-1} - 1) = 2 \cdot (2^k - 1).$$

However, $r = 2^k - 1$ by part (3) of this claim, so equality must hold throughout the above inequalities, thereby proving (4). ◊

Pick $S \subseteq \{0,1\}^r$ containing $\mathbf{0}$ such that $\mathcal{C} = \text{cuboid}(S)$. We prove that $S = \text{cocycle}(PG(k-1,2))$. Denote by A the incidence matrix of S. Notice that A is a column submatrix of $M(\mathcal{C})$, and the column labels of A form a subset of V and a transversal of $\{\{u, f(u)\} : u \in V\}$.

Claim 5. *In A the sum of every two columns modulo 2 is equal to another column.*

Proof of Claim. Pick two columns of A with column labels $u, w \in V$. By Claim 4 (4), in $M(\mathcal{C} \setminus f(u)/u)$, column w is identical to another column v. Notice that $v \in V - \{u, f(u), w, f(w)\}$. By Claim 4 (1), in $M(\mathcal{C}/f(u) \setminus u)$, columns w, v are complementary. Thus, in $M(\mathcal{C})$, columns u, w, v add up to $\mathbf{1}$ modulo 2, implying in turn that columns $u, w, f(v)$ add up to $\mathbf{0}$ modulo 2. We know that columns u, w of $M(\mathcal{C})$ are also present in A, and that exactly one of the columns $v, f(v)$ of $M(\mathcal{C})$ is present in A. As $\mathbf{0} \in S$, A has a zero row, so no three of its columns can add up to $\mathbf{1}$ modulo 2, implying in turn that $f(v)$ must be a column of A instead of v. As a result, in A, columns u, w add up to column $f(v)$ modulo 2, as required. $\Diamond$

We next use Remark 4.4.2 to argue that up to permuting rows and columns, A is the incidence matrix of $\text{cocycle}(PG(k-1,2))$. To this end, denote by $v_0 \in V$ the label of the first column of A. For $j \in \{0,1\}$, denote by I_j the rows of A corresponding to $\{x \in S : x_{v_0} = j\}$. By Claim 2 (3), $|I_0| = |I_1| = 2^{k-1}$. Notice that $\frac{r-1}{2} = 2^{k-1} - 1$ by Claim 4 (3). Label the columns of A other than v_0 as $v_1, u_1, v_2, u_2, \ldots, v_{\frac{r-1}{2}}, u_{\frac{r-1}{2}}$ where for each $i \in \left[\frac{r-1}{2}\right]$, the sum of columns v_0 and v_i modulo 2 is equal to column u_i – such a labeling exists because of Claim 5. Define matrices A_1, A_2, A_3, A_4:

- A_1 is the $I_1 \times \{v_1, \ldots, v_{\frac{r-1}{2}}\}$ submatrix of A,
- A_2 is the $I_1 \times \{u_1, \ldots, u_{\frac{r-1}{2}}\}$ submatrix of A,
- A_3 is the $I_0 \times \{v_1, \ldots, v_{\frac{r-1}{2}}\}$ submatrix of A,
- A_4 is the $I_0 \times \{u_1, \ldots, u_{\frac{r-1}{2}}\}$ submatrix of A.

Then $A_3 = A_4$ and $A_1 + A_2 = J$. After swapping the labels v_i and $u_i, i \in \left[\frac{r-1}{2}\right]$, if necessary, we may assume that A_1 has a zero row. Notice further that as $\mathbf{0} \in S$ and $A_3 = A_4$, the matrix A_3 also has a zero row. As a result, by Claim 4 (4), up to permuting rows and columns, the following three matrices are equal: A_1, A_3, and the incidence matrix of $\text{cocycle}(PG(k-2,2))$.

For the rest of the proof, we work with the projective geometry $PG(k-2,2)$ whose labeling agrees with the column labels of A_3, that is, the cocycles of the labeled $PG(k-2,2)$ are the rows of A_3.

Claim 6. *Up to permuting rows, A_1 and A_3 are equal.*

Proof of Claim. This is obviously true if $k = 2$. We may therefore assume that $k \geq 3$. It suffices to show that every row of A_1 is equal to some row of A_3, because the two matrices are already equal up to permuting rows and columns. Pick a row χ_D of A_1 for some $D \subseteq \{v_1, \dots, v_{\frac{r-1}{2}}\}$. We need to show that D is a cocycle of (the labeled) $PG(k-2, 2)$. Pick a triangle $\{v_i, v_j, v_k\}$ of $PG(k-2, 2)$, that is, the corresponding columns of A_3 add up to zero modulo 2. Consider now the columns v_i, v_j of A. By Claim 5, the sum of these two columns modulo 2 is another column of A. This column is either v_k or u_k, and in fact since A_1 has a zero row, it must be v_k. As a result, columns v_i, v_j, v_k of A_1 also add up to zero modulo 2, implying in turn that $|D \cap \{v_i, v_j, v_k\}|$ is even. Thus, D intersects every triangle of $PG(k-2, 2)$ an even number of times, so by Proposition 4.3.3 (iii), D intersects every cycle of $PG(k-2, 2)$ an even number of times, implying in turn that D is a cocycle of $PG(k-2, 2)$, as required. $\Diamond$

We may therefore assume that $A_1 = A_3$, implying in turn that $A_1 = A_3 = A_4$ and $A_2 = J - A_1$. As A_1 is the incidence matrix of cocycle($PG(k-2, 2)$), it follows from Remark 4.4.2 that A is the incidence matrix of cocycle($PG(k-1, 2)$), so S = cocycle($PG(k-1, 2)$). As $\mathcal{C}$ = cuboid(S), and as $r = 2^k - 1$ by Claim 4 (3), we have finished the proof of Lemma 4.4.4. $\square$

It is worth pointing out that the assumption $|\mathcal{C}| = r + 1$ in Lemma 4.4.4 can be removed without affecting the conclusion, but this comes at the expense of a much longer proof of Claim 4, parts (3) and (4), one that requires the notion of *binary clutters*.

Proof of Theorem 4.1.7 ($\Rightarrow$)

Let $\mathcal{C}$ be a clean tangled clutter over ground set V whose setcore has a simplicial convex hull. By Theorem 4.2.11, $\mathcal{C}$ has a unique fractional packing y of value two. We shall prove by induction on $|V| \geq 2$ that

> ($\star$) there is an integer $k \geq 1$ such that y is $\frac{1}{2^{k-1}}$-integral, rank($\mathcal{C}$) = $2^k - 1$ and support(y) is a duplication of
>
> $$\text{cuboid}(\text{cocycle}(PG(k-1, 2))).$$

For the base case $|V| = 2$, as $\mathcal{C}$ is tangled, it must consist of two members of size one each, so ($\star$) holds for $k = 1$. For the induction step, assume that $|V| \geq 3$. Let $r :=$ rank($\mathcal{C}$) and $S :=$ setcore($\mathcal{C}$) $\subseteq \{0,1\}^r$. By Theorem 4.1.5 (iii) and our assumption, conv(S) is a full-dimensional simplex, implying in turn that $|S| = r + 1$. Let $G := G(\mathcal{C})$, and for each $i \in [r]$, let $\{U_i, V_i\}$ be the bipartition of the i^{th} connected component of G. As support(y) $\subseteq$ core($\mathcal{C}$), Remark 4.2.4 implies Claim 1 below:

Claim 1. *For each $C \in$ support(y) and $i \in [r]$, $C \cap (U_i \cup V_i)$ is either U_i or V_i.*

Claim 2. *If $r = 1$, then ($\star$) holds for $k = 1$.*

Proof of Claim. Assume that $r = 1$. Then support$(y) \subseteq \{U_1, V_1\}$ by Claim 1, and as support(y) contains a fractional packing of $\mathcal{C}$ of value two, we must have that support$(y) = \{U_1, V_1\}$, and the claim follows. ♢

We may therefore assume that $r \geq 2$.

Claim 3. *The following statements hold:*

(1) $|\text{support}(y)| = r + 1$*,*

(2) support(y) *has a unique fractional packing of value two,*

(3) the elements in each of $U_1, V_1, \dots, U_r, V_r$ *are duplicates in* support(y)*,*

(4) for each $i \in [r]$*, if* $u \in U_i$ *and* $v \in V_i$*, then* $\{u, v\}$ *is a transversal of* support(y)*, and*

(5) for each $i \in [r]$*,* support$(y) \backslash U_i / V_i$ *(resp.* support$(y) / U_i \backslash V_i$*) is a duplication of the cuboid of the cocycle space of a projective geometry.*

Proof of Claim. **(1)** Since y is the unique fractional packing of $\mathcal{C}$ of value two, we have core$(\mathcal{C})$ = support(y). Subsequently, $|\text{support}(y)| = |\text{core}(\mathcal{C})\,| = |S| = r + 1$. **(2)** is obvious, and **(3)** and **(4)** follow from Claim 1. **(5)** By Lemma 4.4.1, the minor $\mathcal{C} \setminus U_i / V_i$ is a clean tangled clutter with a unique fractional packing z of value two, and support(z) = support$(y) \setminus U_i / V_i$. Our induction hypothesis applied to $\mathcal{C} \setminus U_i / V_i$ implies that support(z), which is equal to support$(y) \setminus U_i / V_i$, is a duplication of the cuboid of the cocycle space of a projective geometry, as required. ♢

We may therefore apply Lemma 4.4.4 to support(y) to conclude that for some integer $k \geq 2$, $r = 2^k - 1$ and support(y) is a duplication of

$$\text{cuboid}(\text{cocycle}(PG(k-1, 2))).$$

It follows from Theorem 4.3.4 that y assigns $\frac{1}{2^{k-1}}$ to the members of support(y), so $(\star)$ holds. This completes the induction step.

We have shown that $(\star)$ holds. As a consequence, core$(\mathcal{C})$ = support(y) is a duplication of the cuboid of cocycle$(PG(k-1, 2)) \subseteq \{0, 1\}^r$. The uniqueness of the setcore (Theorem 4.1.5 (i)) implies that setcore$(\mathcal{C})$ = cocycle$(PG(k-1, 2))$, thereby finishing the proof of Theorem 4.1.7 $(\Rightarrow)$. □

Binary clutters and an application

A clutter $\mathcal{C}$ is *binary* if the symmetric difference of any three members contains a member [18]. Observe that if a clutter is binary, then so is every duplication of it. It is known that $\mathcal{C}$ is a binary clutter if, and only if, $|C \cap B| \equiv 1 \pmod 2$ for all $C \in \mathcal{C}, B \in b(\mathcal{C})$ [18]. In particular, a clutter is binary if and only if its blocker is binary. Observe that the deltas, extended odd holes and their blockers are not binary. If a clutter is binary, so is every minor of it [22]. Subsequently,

Remark 4.4.5. *Every binary clutter is clean.*

Examples include the clutter of minimal T-joins of a graft, and the clutter of odd circuits of a signed graph (the ground set in each case is the edge set of the underlying graph) [12]. Another class of binary clutters comes from affine binary spaces.

Remark 4.4.6. *Take an integer $n \geq 1$ and a set $S \subseteq \{0,1\}^n$. Then S is an affine binary space if, and only if,* $\text{cuboid}(S)$ *is a binary clutter.*

We are now ready to prove the following appealing consequence of Theorem 4.1.7:

Theorem 4.4.7. *Take an integer $n \geq 1$ and a set $S \subseteq \{0,1\}^n$ whose convex hull is a simplex containing $\frac{1}{2} \cdot \mathbf{1}$ in its relative interior. Then exactly one of the following statements holds:*

- $\text{cuboid}(S)$ *has a delta or the blocker of an extended odd hole minor, or*
- *S is a duplication of the cocycle space of a projective geometry over the two-element field.*

Proof. Let $\mathcal{C} := \text{cuboid}(S)$. If S is a duplication of the cocycle space of a projective geometry, then up to twisting, S is a binary space, so $\mathcal{C}$ is a binary clutter by Remark 4.4.6, implying in turn that it is clean by Remark 4.4.5. Conversely, assume that $\mathcal{C}$ is clean. As $\text{conv}(S)$ is a simplex containing $\frac{1}{2} \cdot \mathbf{1}$ in its relative interior,

- the points in S do not all agree on a coordinate, so $\mathcal{C}$ is tangled, and
- by Remark 4.2.5 on the connection between $\text{conv}(S)$ and fractional packings of $\mathcal{C}$, $\mathcal{C}$ must have a unique fractional packing of value two, one whose support is $\mathcal{C}$.

In particular, $\mathcal{C} = \text{core}(\mathcal{C})$, so S is a duplication of $\text{setcore}(\mathcal{C})$. As $\text{conv}(S)$ is a simplex, so is $\text{conv}(\text{setcore}(\mathcal{C}))$, so by Theorem 4.1.7, $\text{setcore}(\mathcal{C})$ is isomorphic to the cocycle space of a projective geometry, implying in turn that S is a duplication of the cocycle space of a projective geometry, as required. □

4.5 Finding the Fano plane as a minor

In this section, after presenting a few ingredients, we prove Theorem 4.1.8, and then prove a consequence of the result.

Monochromatic covers in clean tangled clutters

Let $\mathcal{C}$ be a clean tangled clutter. A cover is *monochromatic* if it is monochromatic in some (proper) bicoloring of the bipartite graph $G(\mathcal{C})$. In this subsection, we prove a lemma on monochromatic minimal covers in clean tangled clutters. We need the following result from Mathematical Logic:

Proposition 4.5.1 ([21]). *Take an integer $r \geq 1$ and a set $S \subseteq \{0,1\}^r$. Pick disjoint subsets $I, J \subseteq [r]$ and disjoint subsets $I', J' \subseteq [r]$ such that the following inequalities are valid S:*

$$\sum_{i\in I} x_i + \sum_{j\in J}(1-x_j) \geq 1$$
$$\sum_{i\in I'} x_i + \sum_{j\in J'}(1-x_j) \geq 1$$

If $k \in I \cap J'$, then the following inequality is also valid for S:

$$\sum_{i\in(I\cup I')-\{k\}} x_i + \sum_{j\in(J\cup J')-\{k\}} (1-x_j) \geq 1.$$

Proof. We leave the proof as an exercise for the reader. □

Proposition 4.5.1 is known as the *Resolution Principle* and the derived inequality is referred to as the *resolvent* of the other two inequalities. We use this remark to prove the following, a key ingredient needed for the proof of Theorem 4.1.8.

Theorem 4.5.2. *Let $\mathcal{C}$ be a clean tangled clutter over ground set V of rank r, and for each $i \in [r]$, denote by $\{U_i, V_i\}$ the bipartition of the i^{th} connected component of $G := G(\mathcal{C})$. Suppose for some integer $k \in [r]$ that $V_1 \cup \cdots \cup V_k$ is a cover of $\mathcal{C}$. Then $k \geq 3$. Moreover, if $k = 3$, then $V_1 \cup V_2 \cup V_3$ contains a minimal cover of cardinality three picking exactly one element from each $V_i, i \in [3]$.*

Proof. Let $S := \text{setcore}(\mathcal{C} : U_1, V_1; U_2, V_2; \ldots; U_r, V_r)$. Since $V_1 \cup \cdots \cup V_k$ is a cover of $\mathcal{C}$, it is also a cover of core($\mathcal{C}$), so every member of core($\mathcal{C}$) contains at least one of $V_1, \ldots, V_k$, by Remark 4.2.4. In particular, the inequality $x_1 + \cdots + x_k \geq 1$ is valid for conv(S). As $\frac{1}{2} \cdot \mathbf{1}$ lies in the interior of conv(S) by Theorem 4.1.5 (iii), it follows that $k \geq 3$.

Assume that $k = 3$. Let B be a minimum cardinality cover contained in $V_1 \cup V_2 \cup V_3$. What we just showed implies that $B \cap V_i \neq \emptyset$ for $i \in [3]$. We claim that $|B \cap V_i| = 1$ for each $i \in [3]$, thereby finishing the proof. Suppose otherwise. We may assume that $|B \cap V_3| \geq 2$. Let $I := B - V_3$, $J := V - (I \cup U_3 \cup V_3)$, and $\mathcal{C}' := \mathcal{C} \setminus I/J$. Note that $\mathcal{C}'$ is a clean clutter and has ground set $U_3 \cup V_3$. Notice further that every edge of $G[U_3 \cup V_3]$ gives a cardinality-two cover of $\mathcal{C}'$.

Case 1: $\tau(\mathcal{C}') = 2$. In this case, $\mathcal{C}'$ is a tangled clutter where $G[U_3 \cup V_3] \subseteq G(\mathcal{C}')$. In particular, $G(\mathcal{C}')$ is a connected bipartite graph whose bipartition inevitably is $\{U_3, V_3\}$. Thus, rank$(\mathcal{C}') = 1$, so core$(\mathcal{C}') = \{U_3, V_3\}$ by Theorem 4.2.10 (i). However, as B is a cover of $\mathcal{C}$, $B - I = B \cap V_3$ is a cover of $\mathcal{C}'$, a contradiction as $B \cap V_3$ is disjoint from $U_3 \in$ core$(\mathcal{C}') \subseteq \mathcal{C}'$.

Case 2: $\tau(\mathcal{C}') \leq 1$. That is, there is a minimal cover D of $\mathcal{C}$ such that $D \cap J = \emptyset$ and $|D - I| \leq 1$. As $D \cap I \subseteq I = B - V_3 \subsetneq B$, and B is a minimal cover of $\mathcal{C}$, it follows that $D \cap I$ is not a cover of $\mathcal{C}$, so $D - I \neq \emptyset$. Thus, $|D - I| = 1$. Let u be the element in $D - I \subseteq U_3 \cup V_3$.

Case 2.1: $u \in U_3$. In this case, $V_1 \cup V_2 \cup U_3$ is a cover of $\mathcal{C}$, implying that the inequality $x_1 + x_2 + (1 - x_3) \geq 1$ is valid for S. However, $V_1 \cup V_2 \cup V_3$ is also a cover of $\mathcal{C}$, so $x_1 + x_2 + x_3 \geq 1$ is valid for S, too. By applying the Resolution Principle, Proposition 4.5.1, we get that $x_1 + x_2 \geq 1$ is also valid for S. However, $\frac{1}{2} \cdot \mathbf{1}$ lies in the interior of conv(S) by Theorem 4.1.5 (iii), a contradiction.

Case 2.2: $u \in V_3$. In this case,

$$|D| = |D \cap I| + |D - I| = |B - V_3| + 1 < |B - V_3| + |B \cap V_3| = |B|,$$

where the strict inequality follows from our contrary assumption that $|B \cap V_3| \geq 2$. However, $|D| < |B|$ contradicts our minimal choice of B as the minimum cover of $\mathcal{C}$ contained in $V_1 \cup V_2 \cup V_3$.

We obtained a contradiction in each case, as desired. □

A lemma for finding an $\mathbb{L}_7$ minor

Recall that

$$\mathbb{L}_7 = \{\{1,2,3\},\{1,4,5\},\{1,6,7\},\{2,5,6\},\{2,4,7\},\{3,4,6\},\{3,5,7\}\}$$

and $b(\mathbb{L}_7) = \mathbb{L}_7$. This clutter enjoys a lot of symmetries. $\mathbb{L}_7$ has an automorphism mapping every element to every other element, and an automorphism mapping every member to every other member. These facts are crucial throughout this subsection.

Remark 4.5.3. *Let $G = (V, E)$ be a connected, bipartite graph with bipartition $\{U, U'\}$ where $U, U' \neq \emptyset$. Assume that there exists a subset $X \subseteq U'$ such that $2 \leq |X| \leq 3$, and there is no proper vertex-induced subgraph that is connected and contains X. Then G is a tree whose leaves are in X.*

Proof. By our minimality assumption, every vertex in $V - X$ is a cut-vertex of G separating at least two vertices in X. We claim that G is a tree. Suppose otherwise. Then there is a circuit $C \subseteq V$. For every vertex $v \in C$, there is a vertex $g(v) \in X$ such that

- if $v \in X$, then $g(v) = v$,

- otherwise, $g(v)$ is a vertex of X such that every path between it and $C-\{v\}$ includes v.

Notice that if v, v' are distinct vertices of C, then $g(v) \neq g(v')$. In particular, $|X| \geq |C|$, implying in turn that $|C| = 3$, a contradiction as G is bipartite. Thus G is a tree. It is immediate from our minimality assumption that every leaf of G belongs to X. □

We are now ready to prove the following lemma, the workhorse for the proof of Theorem 4.1.8:

Lemma 4.5.4. *Let $\mathcal{C}$ be a clean tangled clutter, where the following statements hold:*

(a) $\mathcal{C}$ has rank 7, and for each $i \in [7]$, the i^{th} connected component of $G(\mathcal{C})$ has bipartition $\{U_i, V_i\}$.

(b) For each $L \in \mathbb{L}_7$, $\bigcup_{i\notin L} U_i \cup \bigcup_{j\in L} V_j$ contains a member of $\mathcal{C}$.

(c) For all $L \in \mathbb{L}_7$ but at most one, $\bigcup_{j\in L} V_j$ is a cover of $\mathcal{C}$.

Then $\mathcal{C}$ has an $\mathbb{L}_7$ minor.

(In (b), $\bigcup_{i\notin L} U_i \cup \bigcup_{j\in L} V_j$ for each $L \in \mathbb{L}_7$ must in fact be a member by Remark 4.2.4; but the proof is easier to read given the current version of (b).)

Proof. Let $G := G(\mathcal{C})$.

Claim 1. *Take a subset $L \subseteq [7]$ such that $|L| \leq 3$ and $\bigcup_{i\in L} V_i$ is a cover. Then $L \in \mathbb{L}_7$. Moreover, $\bigcup_{i\in L} V_i$ contains a minimal cover of cardinality three picking one element from each $V_i, i \in L$.*

Proof of Claim. As (b) holds, $\bigcup_{i\in L} V_i$ intersects each $\bigcup_{i\notin L} U_i \cup \bigcup_{j\in L} V_j, L \in \mathbb{L}_7$, implying in turn that L is a cover of $\mathbb{L}_7$. As $b(\mathbb{L}_7) = \mathbb{L}_7$ and $|L| \leq 3$, it follows that $L \in \mathbb{L}_7$. The second part follows from Theorem 4.5.2. ◊

Claim 2. *For each $L \in \mathbb{L}_7$, $\bigcup_{i\in L} V_i$ is a cover.*

Proof of Claim. We may assume because of (c) that for each $L \in \mathbb{L}_7 - \{\{3, 5, 7\}\}$, $\bigcup_{i\in L} V_i$ contains a minimal cover B_L; we may assume by Claim 1 that B_L has cardinality three and picks one element from each $V_i, i \in L$. It remains to prove that $V_3 \cup V_5 \cup V_7$ is a cover. Suppose otherwise. Let $\mathcal{C}' := \mathcal{C} \setminus (V_5 \cup V_7)/(U_5 \cup U_7)$.

Assume in the first case that $\tau(\mathcal{C}') \leq 1$. That is, there is a minimal cover $D \in b(\mathcal{C})$ such that $D \cap (U_5 \cup U_7) = \emptyset$ and $|D - (V_5 \cup V_7)| \leq 1$. It follows from Claim 1 that $D - (V_5 \cup V_7) = \{u\}$ for some $u \in V_3 \cup U_1 \cup U_2 \cup U_3 \cup U_4 \cup U_6$. Our contrary assumption tells us that $u \notin V_3$. But then D is disjoint from one of

$$\bigcup_{i\notin L} U_i \cup \bigcup_{j\in L} V_j, \quad L = \{1, 2, 3\}, \{3, 4, 6\},$$

a contradiction to (b).

Assume in the remaining case that $\tau(\mathcal{C}') \geq 2$. Then $\mathcal{C}'$ is clean tangled, and $G' := G(\mathcal{C}')$ has $G \setminus (U_5 \cup V_5 \cup U_7 \cup V_7)$ as a subgraph. Then G' is a bipartite graph where for each $i \in \{1,2,3,4,6\}$, $G'[U_i \cup V_i]$ is connected and has bipartition $\{U_i, V_i\}$. Observe that for $L = \{1,4,5\}, \{2,4,7\}, \{2,5,6\}, \{1,6,7\}$, the set $B_L - (V_5 \cup V_7)$ is a cardinality-two cover, and therefore a minimum cover, of $\mathcal{C}'$. As a consequence, G' has an edge between V_1, V_4, an edge between V_4, V_2, an edge between V_2, V_6, and an edge between V_6, V_1. Let $U := U_1 \cup U_2 \cup V_4 \cup V_6$ and $U' := V_1 \cup V_2 \cup U_4 \cup U_6$. Then $G'[U \cup U']$ is connected and has bipartition $\{U, U'\}$. Since $G'[U_3 \cup V_3]$ is also connected, G' has at most two connected components. It therefore follows from Theorem 4.2.10 (i)-(ii) that either

$$U \cup U_3, U' \cup V_3 \in \mathcal{C}' \quad \text{or} \quad U \cup V_3, U' \cup U_3 \in \mathcal{C}'.$$

Observe that $B_L - (V_5 \cup V_7) = B_L$ is a cover of $\mathcal{C}'$ for $L = \{1,2,3\}, \{3,4,6\}$. However, $B_{\{1,2,3\}} \cap (U \cup U_3) = \emptyset$ and $B_{\{3,4,6\}} \cap (U' \cup U_3) = \emptyset$, a contradiction.

As a result, $V_3 \cup V_5 \cup V_7$ is a cover, as claimed. ◊

We may assume that $\mathcal{C}$ is contraction minimal with respect to being tangled and satisfying (a)-(c). By Claims 1 and 2, for each $L \in \mathbb{L}_7$, there exists a minimal cover $B_L \in b(\mathcal{C})$ of cardinality three picking one element from each $V_i, i \in L$. For each $i \in [7]$, let

$$X_i := V_i \cap \left(\bigcup_{L \in \mathbb{L}_7} B_L \right);$$

notice that $1 \leq |X_i| \leq 3$.

Claim 3. *For each $i \in [7]$, either $|X_i| = 1$ and $|U_i| = |V_i| = 1$, or $2 \leq |X_i| \leq 3$ and $G[U_i \cup V_i]$ is a tree whose leaves are contained in X_i.*

Proof of Claim. Let W be a subset of V such that (1) $W \subseteq U_i \cup V_i$, (2) $X_i \subseteq W$, (3) $|W| \geq 2$, (4) $G[W]$ is connected, and (5) W is minimal subject to (1)-(4). Let $\{U_i', V_i'\}$ be the bipartition of $G[W]$ where $U_i' \subseteq U_i$ and $X_i \subseteq V_i' \subseteq V_i$. Notice that if $|X_i| = 1$ then $|U_i'| = |V_i'| = 1$, and if $2 \leq |X_i| \leq 3$ then $G[W]$ must be a tree whose leaves are contained in X_i by Remark 4.5.3. Let $I := (U_i \cup V_i) - (U_i' \cup V_i')$. Notice that $\mathcal{C}/I$ is clean and tangled, and satisfies (a) and (b). Moreover, since $B_L \cap I = \emptyset$ for each $L \in \mathbb{L}_7$, $\mathcal{C}/I$ also satisfies (c). Our minimal choice of $\mathcal{C}$ implies that $I = \emptyset$, so $U_i' = U_i$ and $V_i' = V_i$, thereby finishing the proof of the claim. ◊

Claim 4. *For each $i \in [7]$, $|X_i| = 1$ and $|U_i| = |V_i| = 1$.*

Proof of Claim. Suppose otherwise. We may assume that $G[U_1 \cup V_1]$ is not an edge. It then follows from Claim 3 that $2 \leq |X_1| \leq 3$ and $G[U_1 \cup V_1]$ is a tree whose leaves are contained in X_1. Pick a leaf u of the tree $G[U_1 \cup V_1]$ that belongs to exactly one of $B_{\{1,2,3\}}, B_{\{1,4,5\}}, B_{\{1,6,7\}}$, and let $\mathcal{C}' := \mathcal{C}/u$. Since u is a leaf of $G[U_1 \cup V_1]$, $\mathcal{C}'$ is clean and tangled, and satisfies (a) and (b). Moreover, as u belongs to exactly one of $(B_L : L \in \mathbb{L}_7)$, $\mathcal{C}'$ also satisfies (c), a contradiction to the minimality of $\mathcal{C}$. ◊

Let $\mathcal{C}' := \mathcal{C}/(U_1 \cup \cdots \cup U_7)$.

Claim 5. $\mathcal{C}' \cong \mathbb{L}_7$.

Proof of Claim. We know that $B_L \in b(\mathcal{C}')$ for each $L \in \mathbb{L}_7$, and that by Claim 1, these are the only minimal covers of $\mathcal{C}'$ of cardinality at most three. After a possible relabeling of its elements, we may assume that $\mathcal{C}'$ has ground set $[7]$, and that $B_L = L$ for each $L \in \mathbb{L}_7$. We claim that $b(\mathcal{C}') = \mathbb{L}_7$. Suppose otherwise. Then $b(\mathcal{C}')$ has a member B of cardinality at least four. As $\mathbb{L}_7 \subseteq b(\mathcal{C}')$, it follows that $|B| = 4$ and $B = [7] - L$ for some $L \in \mathbb{L}_7$. However, B is also a minimal cover of $\mathcal{C}$ that is disjoint from $\bigcup_{i \notin L} U_i \cup \bigcup_{j \in L} V_j$, a contradiction to (b). As a result, $b(\mathcal{C}') = \mathbb{L}_7$, so $\mathcal{C}' = b(\mathbb{L}_7) = \mathbb{L}_7$, as claimed. ♢

As a result, $\mathcal{C}'$ has an $\mathbb{L}_7$ minor, thereby finishing the proof of Lemma 4.5.4. □

Proof of Theorem 4.1.8

Let us start with the following remark about the cocycle space of the Fano matroid:

Remark 4.5.5. cuboid(cocycle($PG(2,2)$)) *is, after a possible relabeling, a clutter over ground set* $\{1, 2, \ldots, 7, \bar{1}, \bar{2}, \ldots, \bar{7}\}$ *satisfying the following statements:*

- *the members are* $\{\bar{i} : i \in [7]\}$ *and* $\{i : i \notin L\} \cup \{\bar{j} : j \in L\}$ *for all* $L \in \mathbb{L}_7$*,*
- *the cardinality-three minimal covers are*

$$\{\bar{i}, \bar{j}, \bar{k}\}, \{\bar{i}, j, k\}, \{i, \bar{j}, k\}, \{i, j, \bar{k}\}$$

  for all $\{i, j, k\} \in \mathbb{L}_7$*,*
- *every cardinality-three minimal cover is contained in exactly two members.*

Let $S :=$ cocycle($PG(2,2)$). As S is a binary space, it follows from Remark 4.3.1 that $S \triangle p = S$ for every point $p \in S$. In particular, every member of cuboid(S) can be treated as the first member $\{\bar{i} : i \in [7]\}$ above.

Proposition 4.5.6. *Let* $\mathcal{C}$ *be a clean tangled clutter over ground set* V *that has a unique fractional packing of value two, and of rank seven. Then* $\mathcal{C}$ *has an* $\mathbb{L}_7$ *minor.*

Proof. Let $G := G(\mathcal{C})$, and for each $i \in [7]$, let $\{U_i, V_i\}$ be the bipartition of the i^{th} connected component of G. Let y be the fractional packing of $\mathcal{C}$ of value two. As $\mathcal{C}$ has rank seven, it follows from Theorem 4.1.7 that support(y) is a duplication of cuboid(cocycle($PG(2,2)$)). As support(y) $\subseteq$ core($\mathcal{C}$), it follows from Remark 4.2.4 and Remark 4.5.5 that, after possibly relabeling and swapping $U_i, V_i, i \in [7]$, the following sets are the members of support(y):

$$\bigcup_{j=1}^{7} V_j \quad \text{and} \quad \bigcup_{i \notin L} U_i \cup \bigcup_{j \in L} V_j \quad \forall L \in \mathbb{L}_7.$$

A subset $B \subseteq V$ is a *special cover of* $\mathcal{C}$ if it is a monochromatic minimal cover intersecting at most three connected components of G.

Claim 1. *If B is a special cover of $\mathcal{C}$, then*

- *there is a unique $L \in \mathbb{L}_7$ such that $B \cap (U_i \cup V_i) \neq \emptyset$ for each $i \in L$,*
- *$\{i \in L : B \cap U_i \neq \emptyset\}$ has even cardinality, and*
- *B is contained in exactly two members of* support(y).

Proof of Claim. This follows immediately from Remark 4.5.5. $\Diamond$

Given a special cover B, we refer to L from Claim 1 as the *Fano line corresponding to* B, and to $\{i \in L : B \cap U_i \neq \emptyset\}$ as the *trace of* B.

Claim 2. *For every Fano line $L \in \mathbb{L}_7$, there are three corresponding special covers with pairwise different traces.*

Proof of Claim. Suppose otherwise. We may assume by symmetry between the members of $\mathbb{L}_7$ that $L = \{1,2,3\}$. By Claim 1, every special cover corresponding to L has trace $\emptyset, \{1,2\}, \{1,3\}$ or $\{2,3\}$. We may assume by symmetry between the members of cuboid(cocycle($PG(2,2)$)) that every special cover corresponding to line L, if any, has trace $\{1,2\}$ or $\{1,3\}$. Let $\mathcal{C}' := \mathcal{C} \setminus V_1/U_1$ and $G' := G(\mathcal{C}')$. By Lemma 4.4.1, $\mathcal{C}'$ is clean and tangled and has a unique fractional packing of value two, and given that z is the fractional packing of $\mathcal{C}'$ of value two, support(z) = support(y) $\setminus V_1/U_1$. In particular, support(z) is a duplication of cuboid(cocycle($PG(1,2)$)) by Remark 4.4.3. Theorem 4.1.7 applied to $\mathcal{C}'$ now tells us that

$$\text{rank}(\mathcal{C}') = 2^2 - 1 = 3.$$

Observe that for $i \in [7] - \{1\}$, $G[U_i \cup V_i] \subseteq G'[U_i \cup V_i]$, so $G'[U_i \cup V_i]$ is connected. Let us refer to the edges of G' not contained in any $G'[U_i \cup V_i], i \in [7] - \{1\}$ as *crossing* edges. We claim that

> ($\star$) for every crossing edge $\{u,v\}$, either $\{u,v\} \subseteq U_4 \cup U_5$, $\{u,v\} \subseteq V_4 \cup V_5$, $\{u,v\} \subseteq U_6 \cup U_7$ or $\{u,v\} \subseteq V_6 \cup V_7$.

To this end, pick distinct $i, j \in [7] - \{1\}$ such that $u \in U_i \cup V_i$ and $v \in U_j \cup V_j$. Then $V_1 \cup \{u, v\}$ contains a minimal cover of $\mathcal{C}$, which is inevitably special. It therefore follows from Claim 1 that $\{1, i, j\} \in \mathbb{L}_7$, and either $\{u, v\} \subseteq U_i \cup U_j$ or $\{u, v\} \subseteq V_i \cup V_j$. Since there is no special cover corresponding to line $\{1, 2, 3\}$ and trace either $\emptyset, \{2, 3\}$, it follows that $\{i, j\} = \{4, 5\}$ or $\{6, 7\}$, so $(\star)$ holds. However, $(\star)$ implies that G' has at least four connected components, so $\text{rank}(\mathcal{C}') \geq 4$, a contradiction. $\Diamond$

Claim 3. *There is a member of* support(y) *that contains six special covers corresponding to different Fano lines.*

Proof of Claim. By Claim 2, there are $21 = 7 \times 3$ special covers $B_1, \dots, B_{21}$ such that for distinct $i, j \in [21]$, if B_i and B_j correspond to the same Fano line, then they have different traces. By Claim 1, each $B_i, i \in [21]$ is contained in exactly two members of support(y). As a result, there is a member of support(y) containing at least $\frac{21 \times 2}{8} > 5$ special covers among $B_1, \dots, B_{21}$, as required. $\Diamond$

We may assume that $\bigcup_{j=1}^{7} V_j$ contains six special covers corresponding to different Fano lines. As $\mathcal{C}$ satisfies conditions (a)-(c), we may apply Lemma 4.5.4 to conclude that $\mathcal{C}$ has an $\mathbb{L}_7$ minor, as required. $\square$

We are now ready for the main result of this section:

Proof of Theorem 4.1.8. Let $\mathcal{C}$ be a clean tangled clutter with a unique fractional packing of value two and of rank more than three. Let y be the fractional packing of $\mathcal{C}$ of value two. It then follows from Theorem 4.1.7 that for some integer $k \geq 3$, $\mathcal{C}$ has rank $2^k - 1$, and support(y) is a duplication of cuboid(cocycle$(PG(k-1, 2))$). We prove by induction on $k \geq 3$ that $\mathcal{C}$ has an $\mathbb{L}_7$ minor. The base case $k = 3$ follows from Proposition 4.5.6. For the induction step, assume that $k \geq 4$. Let $\{U, U'\}$ be a connected component of $G(\mathcal{C})$, and let $\mathcal{C}' := \mathcal{C} \setminus U/U'$. By Lemma 4.4.1, $\mathcal{C}'$ is clean tangled and has a unique fractional packing of value two, and if z is the fractional packing of $\mathcal{C}'$ of value two, then support(z) = support$(y) \setminus U/U'$. In particular, support(z) is a duplication of cuboid(cocycle$(PG(k-2, 2))$) by Remark 4.4.3. Thus $\mathcal{C}'$ has rank $2^{k-1} - 1$ by Theorem 4.1.7, so by the induction hypothesis, $\mathcal{C}'$ and therefore $\mathcal{C}$ has an $\mathbb{L}_7$ minor, thereby completing the induction step. This finishes the proof of Theorem 4.1.8. $\square$

Ideal clutters and an application

Theorem 4.1.8 has a geometric consequence; let us elaborate. A clutter $\mathcal{C}$ over ground set V is *ideal* if the associated set covering polyhedron

$$\left\{ x \in \mathbb{R}^V_+ : \sum_{v \in C} x_v \geq 1 \quad C \in \mathcal{C} \right\}$$

is integral [14] (see also [1]). It can be readily checked by the reader that the deltas and $\mathbb{L}_7$ are non-ideal clutters. (In fact, every identically self-blocking clutter different from $\{\{a\}\}$ is non-ideal [3].) It can also be readily checked that every extended odd hole is non-ideal. It is well-known that a clutter is ideal if and only if its blocker is ideal [16, 19]. In particular, the blocker of an extended odd hole is also non-ideal. Moreover, if a clutter is ideal, so is every minor of it [23]. Thus, every ideal clutter is clean.

Theorem 4.5.7. *Let $\mathcal{C}$ be a clean tangled clutter. If* $\text{conv}(\text{setcore}(\mathcal{C}))$ *is a simplex, then at least one of the following statements holds:*

(i) $\text{setcore}(\mathcal{C}) = \{0, 1\}$*, i.e.* $\text{core}(\mathcal{C})$ *consists of two members that partition the ground set,*

(ii) $\text{setcore}(\mathcal{C}) \cong \{000, 110, 101, 011\}$*, i.e.* $\text{core}(\mathcal{C})$ *is a duplication of* Q_6*, or*

(iii) $\mathcal{C}$ *is non-ideal.*

Proof. Let $r := \text{rank}(\mathcal{C})$. If $r > 3$, then $\mathcal{C}$ has an $\mathbb{L}_7$ minor by Theorem 4.1.8, so (iii) holds in particular. Otherwise, $1 \leq r \leq 3$. It follows from Theorem 4.1.7 that $\text{setcore}(\mathcal{C})$ is isomorphic to either $\text{cocycle}(PG(0, 2)) = \{0, 1\}$ or $\text{cocycle}(PG(1, 2)) = \{000, 110, 101, 011\}$, so either (i) or (ii) holds, as required. □

Observe that the statement of Theorem 4.5.7 is geometric while our proof is purely combinatorial, further stressing the synergy between the combinatorics and the geometry of clean tangled clutters. Recently, the authors gave an example of an infinite class of clean tangled clutters (more precisely, *ideal minimally non-packing* clutters with covering number two) that belong to category (ii) of Theorem 4.5.7 [5].

4.6 Future directions for research

Clean tangled clutters were the subject of study in this paper. It was proved that the convex hull of the setcore of every such clutter is a full-dimensional polytope containing the center point of the hypercube in its interior (Theorem 4.1.5). The setcore has a simplicial convex hull if and only if it is the cocycle space of a projective geometry over the two-element field (Theorem 4.1.7). Moreover, if the setcore has a simplicial convex hull, then the clutter has rank at most three or it has an $\mathbb{L}_7$ minor (Theorem 4.1.8).

We conclude the paper with three directions for future research.

Our results expose a fruitful interplay between the combinatorics and the geometry of clean tangled clutters. Further along these lines, and an extension of Theorem 4.1.5, is a strong duality result that holds for such clutters and relates a geometric parameter to a combinatorial parameter [2].

A clutter $\mathcal{C}$ *embeds* $PG(k-2, 2)$ if some subset of $\mathcal{C}$ is a duplication of the cuboid of $\text{cocycle}(PG(k-2, 2))$. This notion was defined in [11]. We conjecture that,

Conjecture 4.6.1. *Every clean tangled clutter embeds a projective geometry over the two-element field.*

This conjecture has an intimate connection to *dyadic* fractional packings of value two in clean tangled clutters; see [9]. Observe that Theorem 4.1.7 proves Conjecture 4.6.1 when the setcore of the clutter has a simplicial convex hull.

The following variant of Conjecture 4.6.1 has also been conjectured:

Conjecture 4.6.2 ([11]). *There exists an integer $\ell \geq 3$ such that every ideal tangled clutter embeds one of $PG(0,2),\dots,PG(\ell-1,2)$.*

This conjecture has an intimate connection to the idealness of *k-wise intersecting clutters* [11]. Observe that Theorem 4.5.7, which is a consequence of Theorem 4.1.8, proves Conjecture 4.6.2 for $\ell = 3$ when the setcore of the clutter has a simplicial convex hull (in fact, $\ell = 2$ suffices here).

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References

[1] Ahmad Abdi. "Ideal clutters". PhD thesis. University of Waterloo, 2018.

[2] Ahmad Abdi and Gérard Cornuéjols. "Clean tangled clutters and a strong duality theorem". To be submitted. 2020.

[3] Ahmad Abdi, Gérard Cornuéjols, and Dabeen Lee. "Identically self-blocking clutters". In: *Integer programming and combinatorial optimization*. Vol. 11480. Lecture Notes in Comput. Sci. Springer, Cham, 2019, pp. 1–12.

[4] Ahmad Abdi, Gérard Cornuéjols, and Dabeen Lee. "Intersecting restrictions in clutters". In: *Combinatorica* (Apr. 2020).

[5] Ahmad Abdi, Gérard Cornuéjols, and Matt Superdock. "A new infinite class of ideal minimally non-packing clutters". In: *Discrete Math.* 344.7 (2021), pp. 112413, 13.

[6] Ahmad Abdi and Bertrand Guenin. "The minimally non-ideal binary clutters with a triangle". In: *Combinatorica* 39.4 (Mar. 2019), pp. 719–752.

[7] Ahmad Abdi and Dabeen Lee. "Deltas, extended odd holes and their blockers". In: *Journal of Combinatorial Theory, Series B* 136 (2019), pp. 193–203.

[8] Ahmad Abdi, Kanstantsin Pashkovich, and Gérard Cornuéjols. "Ideal clutters that do not pack". In: *Mathematics of Operations Research* 43.2 (2017), pp. 533–553.

[9] Ahmad Abdi et al. "Clean clutters and dyadic fractional packings". To be submitted. 2020.

[10] Ahmad Abdi et al. "Cuboids, a class of clutters". In: *Journal of Combinatorial Theory, Series B* 142 (2020), pp. 144–209.

[11] Ahmad Abdi et al. "Idealness of k-wise intersecting families". In: *Lecture Notes in Computer Science* (2020), pp. 1–12.

[12] Gérard Cornuéjols. *Combinatorial optimization: Packing and covering*. Vol. 74. SIAM, 2001.

[13] Gérard Cornuéjols, Bertrand Guenin, and François Margot. "The packing property". In: *Math. Program.* 89.1, Ser. A (2000), pp. 113–126.

[14] Gérard Cornuéjols and Beth Novick. "Ideal $0,1$ matrices". In: *J. Combin. Theory Ser. B* 60.1 (1994), pp. 145–157.

[15] Jack Edmonds and D. R. Fulkerson. "Bottleneck extrema". In: *J. Combinatorial Theory* 8 (1970), pp. 299–306.

[16] D. R. Fulkerson. "Blocking polyhedra". In: *Graph Theory and its Applications (Proc. Advanced Sem., Math. Research Center, Univ. of Wisconsin, Madison, Wis., 1969)*. Academic Press, New York, 1970, pp. 93–112.

[17] J. R. Isbell. "A class of simple games". In: *Duke Math. J.* 25 (1958), pp. 423–439.

[18] Alfred Lehman. "A solution of the Shannon switching game". In: *J. Soc. Indust. Appl. Math.* 12 (1964), pp. 687–725.

[19] Alfred Lehman. "On the width-length inequality". In: *Math. Programming* 16.2 (1979), pp. 245–259.

[20] James Oxley. *Matroid theory*. Second. Vol. 21. Oxford Graduate Texts in Mathematics. Oxford University Press, Oxford, 2011, pp. xiv+684.

[21] J. A. Robinson. "A machine-oriented logic based on the Resolution Principle". In: *Journal of the ACM (JACM)* 12.1 (Jan. 1965), pp. 23–41.

[22] P. D. Seymour. "The forbidden minors of binary clutters". In: *J. London Math. Soc. (2)* 12.3 (1976), pp. 356–360.

[23] P. D. Seymour. "The matroids with the max-flow min-cut property". In: *J. Combinatorial Theory Ser. B* 23.2-3 (1977), pp. 189–222.

Chapter 5

A new infinite class of ideal minimally non-packing clutters

Joint work with Ahmad Abdi and Gérard Cornuéjols.

Discrete Mathematics 344.7 (2021): 112413.

Abstract

The $\tau = 2$ *Conjecture* predicts that every ideal minimally non-packing clutter has covering number two. In the original paper where the conjecture was proposed, in addition to an infinite class of such clutters, thirteen small instances were provided. The construction of the small instances followed an ad-hoc procedure and why it worked has remained a mystery, until now. In this paper, using the theory of *clean tangled clutters*, we identify key structural features about these small instances, in turn leading us to a second infinite class of ideal minimally non-packing clutters with covering number two. Unlike the previous infinite class consisting of *cuboids* with unbounded *rank*, our class is made up of non-cuboids, all with rank three.

5.1 Introduction

Let $\mathcal{C}$ be a clutter over ground set V.[1] The *packing number*, denoted $\nu(\mathcal{C})$, is the maximum number of pairwise disjoint members. The *covering number*, denoted $\tau(\mathcal{C})$, is the minimum cardinality of a cover, i.e. the minimum number of elements needed to intersect every member. We have $\nu(\mathcal{C}) \leq \tau(\mathcal{C})$; this motivates the following standard definitions:

[1]To pick up the pace of the introduction, we have assumed familiarity with standard notions such as clutters, minors, etc. and have postponed their definition to §5.1.

- $\mathcal{C}$ *packs* if $\nu(\mathcal{C}) = \tau(\mathcal{C})$.
- $\mathcal{C}$ has the *packing property* if every minor of $\mathcal{C}$, including $\mathcal{C}$ itself, packs [8].
- $\mathcal{C}$ is *minimally non-packing* if $\mathcal{C}$ does not pack, but every proper minor of $\mathcal{C}$ packs [8].

Observe that a clutter has the packing property if, and only if, it has no minimally non-packing minor. It was proved in [8] that if $\mathcal{C}$ has the packing property, then the set covering polyhedron

$$\left\{ x \in \mathbb{R}^V_+ : \sum_{v \in C} x_v \geq 1 \quad C \in \mathcal{C} \right\}$$

is integral, that is, $\mathcal{C}$ is *ideal* [9]. This implication is a consequence of a powerful theorem of Lehman [15] on the structure of a *minimally non-ideal* clutter, i.e. a non-ideal clutter whose proper minors are ideal. Since the packing property implies idealness, every minimally non-packing clutter is either ideal or minimally non-ideal.

An important conjecture in the area is the *Replication Conjecture* [6], stating that a minimally non-packing clutter cannot have replicated elements. Lehman's theorem verifies the conjecture for the minimally non-ideal clutters (see [8]). It therefore remains to prove that an ideal minimally non-packing clutter cannot admit replicated elements. To solve the remaining case, Cornuéjols, Guenin and Margot made the following stronger conjecture:

Conjecture 5.1.1 ($\tau = 2$ Conjecture [8]). *Every ideal minimally non-packing clutter has covering number two.*

All the examples of ideal minimally non-packing clutters known at the time had covering number two, making the authors of [8] believe the conjecture above. The reader may think that the reason for believing the conjecture is somewhat superficial but, recently, geometric evidence supporting the conjecture was provided in [4].

Given the $\tau = 2$ Conjecture, a natural research direction is to study, give examples of, and characterize ideal minimally non-packing clutters with covering number two. In [8], the authors provided an infinite class of such clutters along with thirteen small examples. The infinite class of ideal minimally non-packing clutters consists of *cuboids*, and in papers [3, 4], by using the theory of cuboids, more than 700 new *small* cuboid examples were generated via a computer program. Twelve of the thirteen small examples [8], however, are not cuboids. Other than the fact that they belong to chains of ideal minimally non-packing clutters starting with Q_6 [3], not much else has been known about them.

In this paper, we use the theory of *clean tangled clutters* to identify key structural properties about the thirteen small examples of ideal minimally non-packing clutters with covering number two, and in general about those with *rank* three. This investigation leads us to a new infinite class of ideal minimally non-packing clutters with covering number two, clutters which share the identified structural properties with the thirteen small examples.

Since the new infinite class of ideal minimally non-packing clutters is easy to describe, and the proof of correctness does not need much overhead knowledge, we present these two things first. Given a clutter $\mathcal{C}$ over ground set V, we define $G(\mathcal{C})$ as the graph with vertices V and edges corresponding to the two-element minimal covers of $\mathcal{C}$.

Theorem 5.1.2. *Let $\mathcal{C}$ be a clutter, and let $G = G(\mathcal{C})$. Assume that*

- *G is bipartite and has exactly 3 connected components,*
- *the first connected component of G has two vertices $1, 2$ and an edge between them,*
- *the second connected component of G has two vertices $3, 4$ and an edge between them,*
- *the third connected component of G is a path on at least four edges, where the first edge is $\{5, 6\}$, the last edge is $\{5', 6'\}$, $5, 5'$ belong to the same part of the bipartition, and $6, 6'$ belong to the other part of the bipartition, and*
- *the minimal covers of $\mathcal{C}$ of cardinality different from two are precisely*

$$\{2, 4, 6\}, \{2, 3, 5\}, \{1, 4, 5'\}, \{1, 3, 6'\} \quad \textit{and} \quad \{3, 5, 6'\}, \{4, 5', 6\}.$$

Then $\mathcal{C}$ is an ideal minimally non-packing clutter. (See Figure 5.1 for an illustration of the graph G.)

After introducing some definitions and a preliminary in §5.1, we prove Theorem 5.1.2 in §5.2. Then we prove in §5.3 that all ideal minimally non-packing clutters with covering number two and *rank three* share certain structural properties – properties that our new examples enjoy. We conclude the paper in §5.4 by describing the thirteen examples of ideal minimally non-packing clutters of [8] that we alluded to, and noting that these examples also enjoy the structural properties discussed in §5.3.

Definitions and a preliminary

Given a finite set V, a *clutter* $\mathcal{C}$ over ground set V is a family of subsets of V, such that $C \not\subseteq C'$ for all $C, C' \in \mathcal{C}$. We refer to elements of V simply as *elements*, and sets in $\mathcal{C}$ as *members*. A *cover* of $\mathcal{C}$ is a subset $B \subseteq V$ with $B \cap C \neq \emptyset$ for all $C \in \mathcal{C}$. A *transversal* of $\mathcal{C}$ is a cover $B \subseteq V$ with $|B \cap C| = 1$ for all $C \in \mathcal{C}$. A cover is *minimal* if it does not contain another cover. The *blocker* $b(\mathcal{C})$ of $\mathcal{C}$ is the clutter over ground set V consisting of the minimal covers of $\mathcal{C}$ [10]. For all clutters $\mathcal{C}$, we have $b(b(\mathcal{C})) = \mathcal{C}$ [10, 13].

For disjoint $I, J \subseteq V$, the *minor* $\mathcal{C} \setminus I/J$ of $\mathcal{C}$ obtained by *deleting* I and *contracting* J is the clutter over ground set $V-(I\cup J)$ consisting of the inclusion-wise minimal sets in $\{C \setminus J : C \in \mathcal{C},\ C \cap I = \emptyset\}$. We say that $\mathcal{C} \setminus I/J$ is a *proper* minor of $\mathcal{C}$ if $I \cup J \neq \emptyset$. Deletions in $\mathcal{C}$ correspond to contractions in $b(\mathcal{C})$, and contractions in $\mathcal{C}$ correspond to deletions in $b(\mathcal{C})$, so that $b(\mathcal{C} \setminus I/J) = b(\mathcal{C})/I \setminus J$ [19].

Recall that a clutter $\mathcal{C}$ is ideal if its set covering polyhedron

$$\left\{x \in \mathbb{R}_+^V : \sum_{v \in C} x_v \geq 1 \quad C \in \mathcal{C}\right\}$$

is integral. Idealness is closed under taking the blocker and minors. That is, $\mathcal{C}$ is ideal if and only if $b(\mathcal{C})$ is ideal [12, 14], and if $\mathcal{C}$ is ideal, then all minors of $\mathcal{C}$ are ideal [20].

We now give some examples of non-ideal clutters. For any integer $n \geq 3$, the clutter Δ_n over ground set $[n] = \{1, \ldots, n\}$ is given by

$$\Delta_n = \{\{1,2\}, \{1,3\}, \ldots, \{1,n\}, \{2,3,\ldots,n\}\}$$

More generally, a *delta* is any clutter whose elements can be relabeled to obtain Δ_n for some n. An *extended odd hole* is any clutter whose elements can be relabeled as $[n]$, for odd $n \geq 5$, to obtain a clutter $\mathcal{C}$ of the form

$$\mathcal{C} = \{\{1,2\}, \{2,3\}, \ldots, \{n-1,n\}, \{n,1\}\} \cup \mathcal{C}'$$

where $\mathcal{C}'$ consists of members with cardinality three or more. Note that the set covering polyhedron of Δ_n has a fractional vertex $\left(\frac{n-2}{n-1}, \frac{1}{n-1}, \ldots, \frac{1}{n-1}\right)$, and the set covering polyhedron of any extended odd hole has a fractional vertex $\left(\frac{1}{2}, \ldots, \frac{1}{2}\right)$. Since a clutter $\mathcal{C}$ is ideal if and only if $b(\mathcal{C})$ is ideal, the blocker of any extended odd hole is also non-ideal. We might also remark that $b(\Delta_n)$ is non-ideal, except that $b(\Delta_n) = \Delta_n$. We need the following result for the proof of Theorem 5.1.2:

Theorem 5.1.3 ([15], see also [18]). *Every minimally non-packing clutter with covering number two is ideal, a delta, or the blocker of an extended odd hole.*

5.2 Validity of our construction

We begin with a few definitions and notions. Let $\mathcal{C}$ be a clutter over ground set V. Recall that $G(\mathcal{C})$ is the graph with vertices V and edges corresponding to the two-element minimal covers of $\mathcal{C}$.

Definition 5.2.1 ([5]). A clutter $\mathcal{C}$ is *tangled* if $\tau(\mathcal{C}) = 2$ and every element of $\mathcal{C}$ is in a cardinality-two cover. That is, $\mathcal{C}$ is tangled if $G(\mathcal{C})$ has no isolated vertex.

Note that for a tangled clutter $\mathcal{C}$, every member of $\mathcal{C}$ is a vertex cover of $G(\mathcal{C})$. (The converse, however, is not true.)

Definition 5.2.2. The *rank* of a tangled clutter $\mathcal{C}$, denoted rank($\mathcal{C}$), is the number of connected components of $G(\mathcal{C})$.

Note that the clutters $\mathcal{C}$ described in Theorem 5.1.2 are tangled, have rank three, and $G(\mathcal{C})$ is bipartite. We will use these properties, along with the following lemma, to show that the clutters described in Theorem 5.1.2 do not pack:

Lemma 5.2.3. *Let $\mathcal{C}$ be a tangled clutter with rank three, and suppose $G(\mathcal{C})$ is bipartite. Denote by $\{U_1, V_1\}$, $\{U_2, V_2\}$, $\{U_3, V_3\}$ the bipartitions of the connected components of $G(\mathcal{C})$. If $V_1 \cup V_2 \cup V_3$, $U_1 \cup U_2 \cup V_3$, $U_1 \cup V_2 \cup U_3$, $V_1 \cup U_2 \cup U_3$ are covers of $\mathcal{C}$, then $\mathcal{C}$ does not pack.*

Proof. Suppose for contradiction that $\mathcal{C}$ packs, so that $\mathcal{C}$ has disjoint members C_1, C_2. Then each edge of $G(\mathcal{C})$ must have exactly one end in each of C_1, C_2, so each C_i respects the bipartition of each connected component of $G(\mathcal{C})$; that is, $C_i \cap (U_j \cup V_j) \in \{U_j, V_j\}$ for $i \in [2]$ and $j \in [3]$. We may assume that $C_1 \cap (U_1 \cup V_1) = U_1$ and so $C_2 \cap (U_1 \cup V_1) = V_1$. We have two cases:

Case 1: $C_1 \cap (U_2 \cup V_2) = U_2$ and so $C_2 \cap (U_2 \cup V_2) = V_2$. As $V_1 \cup V_2 \cup V_3$ is a cover, it follows that $C_1 \cap (U_3 \cup V_3) = V_3$ and so $C_2 \cap (U_3 \cup V_3) = U_3$. But then C_2 is disjoint from the cover $U_1 \cup U_2 \cup V_3$.

Case 2: $C_1 \cap (U_2 \cup V_2) = V_2$ and so $C_2 \cap (U_2 \cup V_2) = U_2$. As $V_1 \cup U_2 \cup U_3$ is a cover, it follows that $C_1 \cap (U_3 \cup V_3) = U_3$ and so $C_2 \cap (U_3 \cup V_3) = V_3$. But then C_2 is disjoint from the cover $U_1 \cup V_2 \cup U_3$.

In both cases we have a contradiction, so $\mathcal{C}$ does not pack. □

We will use the following lemma to show that the clutters described in Theorem 5.1.2 are minimally non-packing. Note that this lemma applies even to clutters $\mathcal{C}$ where $G(\mathcal{C})$ has isolated vertices:

Lemma 5.2.4. *Let $\mathcal{C}$ be a clutter over ground set V, and let $G = G(\mathcal{C})$. Assume that:*

(i) G is a bipartite graph with bipartition $\{U_0, V_0\}$,

(ii) $|\{B \in b(\mathcal{C}) : |B| > 2\}| \leq 1$, and

(iii) if $B \in b(\mathcal{C})$ satisfies $|B| > 2$, then $B = \{u, v, w\}$ where

- *$u \in U_0$ and $\{v, w\} \subseteq V_0$, and*
- *in G, either v, w belong to different connected components, or some neighbor of u is a cut-vertex of G separating v and w.*

Then $\mathcal{C}$ has the packing property.

Proof. We proceed by induction on $|V| \geq 2$. The base case $|V| = 2$ holds trivially, so we may assume $|V| \geq 3$. Also, we may assume $\tau(\mathcal{C}) \geq 2$.

Claim 1. *$\mathcal{C}$ packs.*

Proof of Claim. If $\tau(\mathcal{C}) \geq 3$, then every minimal cover of $\mathcal{C}$ has cardinality at least 3, so $b(\mathcal{C}) = \{\{u, v, w\}\}$, so $\mathcal{C} = \{\{u\}, \{v\}, \{w\}\}$, which clearly packs. Otherwise, $\tau(\mathcal{C}) = 2$. Notice that both U_0, V_0 are covers of $b(\mathcal{C})$, so they each contain members of $\mathcal{C}$, implying in turn that $\mathcal{C}$ has two disjoint members, so $\mathcal{C}$ packs. $\Diamond$

Claim 2. *For each $x \in V$, $\mathcal{C}/x$ has the packing property.*

Proof of Claim. Notice that $\mathcal{C}/x$ satisfies (i)-(iii), so the claim follows from the induction hypothesis. $\Diamond$

Claim 3. *For each $x \in V$ and $N = \{y \in V : \{x, y\} \in b(\mathcal{C})\}$, $\mathcal{C} \setminus x/N$ has the packing property.*

Proof of Claim. Let $\mathcal{C}' = \mathcal{C} \setminus x/N$. If $N \neq \emptyset$, then $\mathcal{C}'$ has the packing property by Claim 2. We may therefore assume that $N = \emptyset$. By the induction hypothesis, it suffices to prove that $\mathcal{C}'$ satisfies (i)-(iii). If $x \notin \{u, v, w\}$, then $\mathcal{C}'$ clearly satisfies (i)-(iii). If $x \in \{v, w\}$, then since $u \in U_0$ and $\{v, w\} \subseteq V_0$, it follows that $\mathcal{C}'$ satisfies (i)-(iii). Otherwise $x = u$. As $N = \emptyset$, we see that v, w belong to different connected components of G, so $\mathcal{C}'$ satisfies (i)-(iii), as required. $\Diamond$

These three claims imply that $\mathcal{C}$ has the packing property. To see this, consider an arbitrary minor $\mathcal{C} \setminus I/J$ of $\mathcal{C}$. If $I = J = \emptyset$, then the minor packs by Claim 1. If $J \neq \emptyset$, then the minor packs by Claim 2. Otherwise, $J = \emptyset$ and $I \neq \emptyset$. If $\tau(\mathcal{C} \setminus I/J) < 2$, then the minor obviously packs. Otherwise, $\tau(\mathcal{C} \setminus I/J) \geq 2$, so by Claim 3, the minor packs. We have completed the induction step, thereby finishing the proof of the lemma. $\square$

We are now ready to prove Theorem 5.1.2, stating that if $\mathcal{C}$ is tangled where $G(\mathcal{C})$ is as illustrated in Figure 5.1, and $\{2, 4, 6\}, \{2, 3, 5\}, \{1, 4, 5'\}, \{1, 3, 6'\}$ and $\{3, 5, 6'\}, \{4, 5', 6\}$ are the minimal covers of cardinality different from two, then $\mathcal{C}$ is ideal minimally non-packing:

Proof of Theorem 5.1.2. Let $\mathcal{C}$ be a clutter satisfying the hypotheses of Theorem 5.1.2. Note that $\mathcal{C}$ is tangled. Let $\{U_3, V_3\}$ be the bipartition of the third connected component of G, where $\{5, 5'\} \subseteq U_3$ and $\{6, 6'\} \subseteq V_3$. (See Figure 5.1 for an illustration of G.)

Claim 1. *$\mathcal{C}$ does not pack.*

Proof of Claim. As $\{2, 4, 6\}, \{2, 3, 5\}, \{1, 4, 5'\}, \{1, 3, 6'\}$ are minimal covers,

$$\{2, 4\} \cup V_3, \quad \{2, 3\} \cup U_3, \quad \{1, 4\} \cup U_3, \quad \{1, 3\} \cup V_3$$

are covers, so $\mathcal{C}$ does not pack by Lemma 5.2.3. $\Diamond$

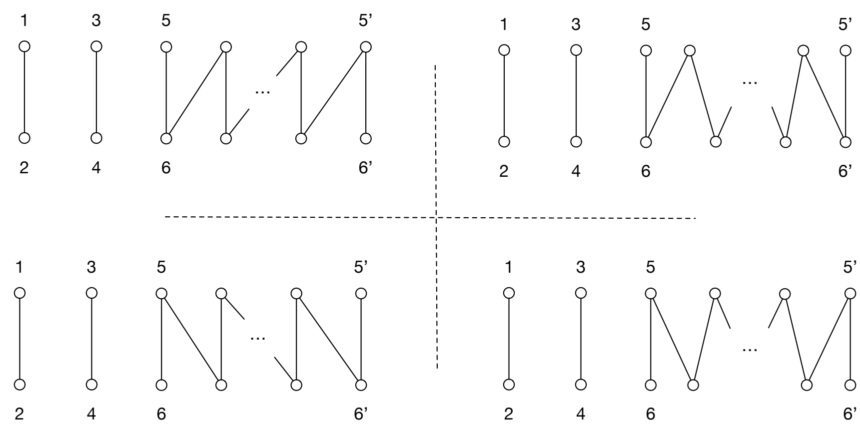

Figure 5.1: The four possibilities for G from Theorem 5.1.2.

In what follows, notice that in our setup, there is symmetry between $1,2$, between $3,4$, between $5,6$, between $5',6'$, and between $\{5,6\},\{5',6'\}$.

Claim 2. *Every proper contraction minor of $\mathcal{C}$ packs.*

Proof of Claim. Choose $I\subseteq V$ such that $\tau(\mathcal{C}/I)\geq 2$. Let $\mathcal{C}'=\mathcal{C}/I$ and notice that $b(\mathcal{C}')=b(\mathcal{C})\setminus I$. As a result, $\tau(\mathcal{C}')\in\{2,3\}$. If $\tau(\mathcal{C}')=3$, then it can be readily checked that $b(\mathcal{C}')$ has at most two members, each of cardinality 3, implying in turn that $\mathcal{C}'$ packs. Otherwise, $\tau(\mathcal{C}')=2$, in which case we need to look for disjoint members in $\mathcal{C}'$. As disjoint members remain disjoint in contraction minors, we may assume that $I=\{x\}$. In what follows, we find disjoint covers in $b(\mathcal{C}')$. Assume in the first case that $x\in\{1,2,3,4,5,6,5',6'\}$. By symmetry, we may assume that $x\notin\{2,4,6,5',6'\}$.

- If $x=1$, then $\{2,3\}\cup V_3$, $\{4\}\cup U_3$ are disjoint covers of $b(\mathcal{C}')$.
- If $x=3$, then $\{1,4\}\cup V_3$, $\{2\}\cup U_3$ are disjoint covers of $b(\mathcal{C}')$.
- Otherwise, $x=5$. Then $\{2,3\}\cup(U_3-\{5\}),\{1,4\}\cup V_3$ are disjoint covers of $b(\mathcal{C}')$.

Assume in the remaining case that $x\in V-\{1,2,3,4,5,6,5',6'\}=(U_3\cup V_3)-\{5,6,5',6'\}$. Notice that deleting x from G disconnects the third connected component, that is, $G\setminus x$ has four connected components with bipartitions $\{\{1\},\{2\}\}$, $\{\{3\},\{4\}\}$, $\{U_3',V_3'\}$, $\{U_3'',V_3''\}$, where $U_3'\cup U_3''=U_3-\{x\}$, $V_3'\cup V_3''=V_3-\{x\}$, $5\in U_3', 6\in V_3', 5'\in U_3''$ and $6'\in V_3''$. Observe now that $\{1,3\}\cup(V_3'\cup U_3''),\{2,4\}\cup(U_3'\cup V_3'')$ are disjoint covers of $b(\mathcal{C}')$.

In each case, we proved that $b(\mathcal{C}')$ has disjoint covers, giving disjoint members of $\mathcal{C}'$ in turn, as desired. $\Diamond$

Claim 3. *Let $I \subseteq V$ be nonempty, such that I is disjoint from*

$$\{1, 2, 3, 4, 5, 6, 5', 6'\}$$

and not a cover of $\mathcal{C}$. Let $N \subseteq V - I$ be a set containing $\big\{y \in V - I : \{x, y\} \in b(\mathcal{C})$ for some $x \in I\big\}$. Then $\mathcal{C} \setminus I/N$ packs.

Proof of Claim. Let $\mathcal{C}' = \mathcal{C} \setminus I/N$, and note that $b(\mathcal{C}') = b(\mathcal{C})/I \setminus N$. As $I \cap \{1, 2, 3, 4, 5, 6, 5', 6'\} = \emptyset$, it follows that $b(\mathcal{C})/I \setminus N = b(\mathcal{C}) \setminus (I \cup N)$, implying that $\mathcal{C}' = \mathcal{C}/(I \cup N)$, so $\mathcal{C}'$ packs by Claim 2. $\Diamond$

Claim 4. *Let $x \in \{1, 2, 3, 4, 5, 6, 5', 6'\}$ and $N = \{y \in V : \{x, y\} \in b(\mathcal{C})\}$. Then $\mathcal{C} \setminus x/N$ has the packing property.*

Proof of Claim. By symmetry, we may assume that $x \notin \{2, 4, 6, 5', 6'\}$. To prove the claim, it suffices to show that $\mathcal{C} \setminus 1/2, \mathcal{C} \setminus 3/4$ and $\mathcal{C} \setminus 5/6$ have the packing property.

Every minimal cover of $\mathcal{C} \setminus 1/2$ has cardinality two, and the graph over vertex set $V - \{1, 2\}$ of the minimal covers of the minor is bipartite with bipartition $\{3\} \cup U_3, \{4\} \cup V_3$, so $\mathcal{C} \setminus 1/2$ has the packing property by Lemma 5.2.4.

Every minimal cover of $\mathcal{C} \setminus 3/4$ also has cardinality two, and the graph over vertex set $V - \{3, 4\}$ of the minimal covers of the minor is bipartite with bipartition $\{1\} \cup U_3, \{2\} \cup V_3$, so once again $\mathcal{C} \setminus 3/4$ has the packing property by Lemma 5.2.4.

Finally, let $\mathcal{C}' = \mathcal{C} \setminus 5/6$, and let G' be the graph over vertex set $V - \{5, 6\}$ whose edges correspond to the cardinality two minimal covers of $\mathcal{C}'$. Notice that G' is a bipartite graph with bipartition $\{1, 3\} \cup (U_3 - \{5\}), \{2, 4\} \cup (V_3 - \{6\})$. Moreover, there is only one minimal cover with cardinality greater than two, namely $\{1, 4, 5'\}$. Furthermore, the neighbor 3 of 4 in G' is a cut-vertex separating $1, 5'$. Thus $\mathcal{C}'$ has the packing property by Lemma 5.2.4. $\Diamond$

These four claims imply that $\mathcal{C}$ is a minimally non-packing clutter. To see this, note first that the clutter does not pack by Claim 1. Let $\mathcal{C} \setminus I/J$ be a proper minor. If $I = \emptyset$, then the minor packs by Claim 2. Otherwise, $I \neq \emptyset$. If $\tau(\mathcal{C} \setminus I/J) < 2$, then the minor clearly packs. Otherwise, $\tau(\mathcal{C} \setminus I/J) \geq 2$. If $I \cap \{1, 2, 3, 4, 5, 6, 5', 6'\} \neq \emptyset$, then the minor packs by Claim 4. Otherwise, I is disjoint from $\{1, 2, 3, 4, 5, 6, 5', 6'\}$. In this case, as $\tau(\mathcal{C} \setminus I/J) \geq 2$, it must be that $J \supseteq \big\{y \in V - I : \{x, y\} \in b(\mathcal{C})$ for some $x \in I\big\}$, so the minor packs by Claim 3. We have exhausted all cases, so every proper minor of $\mathcal{C}$ packs, so $\mathcal{C}$ is a minimally non-packing clutter.

By Theorem 5.1.3, $\mathcal{C}$ is ideal, a delta, or the blocker of an extended odd hole. However, as G is a bipartite graph with at least two connected components, $\mathcal{C}$ must be an ideal clutter, thereby finishing the proof. $\square$

Take an integer $r \geq 1$. Given a set $S \subseteq \{0, 1\}^r$, the *cuboid* of S is the clutter over ground set $[2r]$ whose members have incidence vectors $\{(p_1, 1 - p_1, \ldots, p_r, 1 - p_r) : p \in S\}$. We call a clutter a *cuboid* if it can be obtained from the cuboid of some set S by relabeling elements of the ground set $[2r]$. In other words, a cuboid is a clutter $\mathcal{C}$ whose ground set can be relabeled as $[2r]$ for some integer $r \geq 1$, such that $\{2i-1, 2i\}$ for each $i \in [r]$ is a transversal:

$$|\{1,2\} \cap C| = |\{3,4\} \cap C| = \cdots = |\{2r-1, 2r\} \cap C| = 1 \quad \forall C \in \mathcal{C}.$$

Notice that every member of $\mathcal{C}$ has cardinality r, and that $\{2i-1, 2i\}$ for each $i \in [r]$ is a cover. Thus, if $\mathcal{C}$ has no cover of cardinality one, then $\mathcal{C}$ is a tangled clutter where $G(\mathcal{C})$ has at least the edges $\{1,2\}, \{3,4\}, \ldots, \{2r-1, 2r\}$. Our construction has the promised properties:

Remark 5.2.5. *The clutters $\mathcal{C}$ described in Theorem 5.1.2 have $\tau(\mathcal{C}) = 2$,* $\operatorname{rank}(\mathcal{C}) = 3$*, and are not cuboids.*

Proof. The first two properties are immediate. To see that $\mathcal{C}$ is not a cuboid, note first that $|V| \geq 9$. If $|V|$ is odd, then we are clearly done. Otherwise, $|V|$ is even. We prove that $\mathcal{C}$ has a member of cardinality greater than $|V|/2$, thereby finishing the proof. To this end, let $\{U_3, V_3\}$ be the bipartition of the third connected component of $G(\mathcal{C})$, where $\{5, 5'\} \subseteq U_3$ and $\{6, 6'\} \subseteq V_3$. The third connected component is either a $56'$- or $5'6$-path (see Figure 5.1). In the first case, $\{1,4,5\} \cup V_3$ is a member of $\mathcal{C}$ of cardinality $(|V|+2)/2$, while in the remaining case, $\{1,3,6\} \cup U_3$ is a member of $\mathcal{C}$ of cardinality $(|V|+2)/2$, as claimed. $\square$

That our infinite class of ideal minimally non-packing clutters consists of non-cuboids all with rank three is interesting because the only other known infinite class of such examples, due to [8], is a family of cuboids of unbounded rank (see [3]).

5.3 Structure of ideal minimally non-packing clutters

Our goal in this section is to prove Theorem 5.3.13, that all ideal minimally non-packing clutters with covering number two and rank three share certain structural properties. To describe these properties, we need a few concepts first.

Clean tangled clutters, the core, and the setcore

Let us begin with the following definition:

Definition 5.3.1. A clutter $\mathcal{C}$ is *clean* if no minor of $\mathcal{C}$ is a delta or the blocker of an extended odd hole.

In particular, ideal clutters are clean, so the clutters obtained by our construction are clean. We are particularly interested in clean tangled clutters. To see why the tangled assumption is reasonable, note that in general we have $\tau(\mathcal{C} \setminus v) \leq \tau(\mathcal{C})$ and $\nu(\mathcal{C} \setminus v) \leq \nu(\mathcal{C})$. But if v is in no cover of cardinality $\tau(\mathcal{C})$, then $\tau(\mathcal{C} \setminus v) = \tau(\mathcal{C})$, and in particular $\mathcal{C}$ cannot be minimally non-packing. To summarize:

Remark 5.3.2. *Every ideal minimally non-packing clutter with covering number two is a clean tangled clutter.*

For clean tangled clutters, the graph $G(\mathcal{C})$ of cardinality-two covers takes a nice form:

Theorem 5.3.3 ([2], Remark 6 and Theorem 7)**.** *If $\mathcal{C}$ is a clean tangled clutter, then:*

(i) $G(\mathcal{C})$ is a bipartite graph without isolated vertices ([2], Remark 6).

If, in addition, $G(\mathcal{C})$ is connected and $\{U_1, V_1\}$ is the bipartition of $G(\mathcal{C})$, then:

(ii) Neither of U_1, V_1 is a cover of $\mathcal{C}$ ([2], Theorem 7).

(iii) U_1, V_1 are members of $\mathcal{C}$.

Proof. **(iii)** It follows from (ii) that the complement of U_1, namely V_1, contains a member of $\mathcal{C}$. Since every member of $\mathcal{C}$ is a vertex cover of $G(\mathcal{C})$ and no proper subset of V_1 is a vertex cover, it follows that V_1 itself is a member of $\mathcal{C}$. Similarly, U_1 is a member of $\mathcal{C}$, as required. □

For any clutter $\mathcal{C}$, a *fractional packing* of $\mathcal{C}$ is a feasible point of the following linear program:

$$\begin{array}{ll} \max & \mathbf{1}^\top y \\ \text{s.t.} & \sum (y_C : v \in C \in \mathcal{C}) \leq 1 \quad v \in V \\ & y \geq \mathbf{0}. \end{array}$$

The *value* of a fractional packing y is the value $\mathbf{1}^\top y$ of the linear program. It can be readily verified that by weak duality, the value of any fractional packing is bounded above by $\tau(\mathcal{C})$. The following result was proved recently:

Theorem 5.3.4 ([2], Theorem 3 and [1], Lemma 1.6)**.** *If $\mathcal{C}$ is a clean tangled clutter, then $\mathcal{C}$ has a fractional packing of value two.*

As a result, we may study the structure of the members of $\mathcal{C}$ used in some such fractional packing:

Definition 5.3.5. Let $\mathcal{C}$ be a clean tangled clutter. Then the *core* of $\mathcal{C}$ is the clutter

$$\text{core}(\mathcal{C}) = \{C \in \mathcal{C} : y_C > 0 \text{ for some fractional packing } y \text{ of value two}\}.$$

We use a complementary slackness argument to prove the following:

Lemma 5.3.6. *Let $\mathcal{C}$ be a clean tangled clutter over ground set V. Then every member of* core$(\mathcal{C})$ *is a transversal of the cardinality-two covers of $\mathcal{C}$. Moreover, for every fractional packing y of value two and for every element $v \in V$, we have $\sum (y_C : v \in C \in \mathcal{C}) = 1$.*

Proof. Let $\{u, v\}$ be a cover of $\mathcal{C}$, and let y be a fractional packing of $\mathcal{C}$. Then we have

$$\sum(y_C : C \in \mathcal{C}) \leq \sum(y_C : u \in C \in \mathcal{C}) + \sum(y_C : v \in C \in \mathcal{C}) \leq 2$$

The first inequality follows from the fact that each $C \in \mathcal{C}$ contains either u or v, and the second follows from adding the congestion inequalities for each of u, v. If y has value two, then $\sum(y_C : C \in \mathcal{C}) = 2$, so we have equality above. The first equality implies that if $y_C > 0$, then C contains exactly one of u, v. Therefore, every member of core($\mathcal{C}$) is a transversal of the cardinality-two covers $\{u, v\}$.

The second equality implies that $\sum(y_C : u \in C \in \mathcal{C}) = \sum(u_C : v \in C \in \mathcal{C}) = 1$. Since $\mathcal{C}$ is tangled, every element $v \in V$ appears in some cardinality-two cover of $\mathcal{C}$, so $\sum(y_C : v \in C \in \mathcal{C}) = 1$ for all $v \in V$. □

As a result, since $G(\mathcal{C})$ is bipartite, members of the core must respect the bipartition of each connected component:

Remark 5.3.7. *Let $\mathcal{C}$ be a clean tangled clutter, and let $r =$ rank($\mathcal{C}$). For each $i \in [r]$, denote by $\{U_i, V_i\}$ the bipartition of the i^{th} connected component of $G(\mathcal{C})$. If $C \in$ core($\mathcal{C}$), then $C \cap (U_i \cup V_i) \in \{U_i, V_i\}$ for $i \in [r]$.*

In other words, each $C \in$ core($\mathcal{C}$) is determined by r binary choices; in each connected component, C must contain exactly one of the two parts of the bipartition. This allows a more concise representation of the core:

Definition 5.3.8. Let $\mathcal{C}$ be a clean tangled clutter, and let $r =$ rank($\mathcal{C}$). For each $i \in [r]$, denote by $\{U_i, V_i\}$ the bipartition of the i^{th} connected component of $G(\mathcal{C})$. For each $C \in$ core($\mathcal{C}$), define $p_C \in \{0, 1\}^r$ such that

$$(p_C)_i = \begin{cases} 0 & \text{if } C \cap (U_i \cup V_i) = V_i \\ 1 & \text{if } C \cap (U_i \cup V_i) = U_i \end{cases}$$

The *setcore* of $\mathcal{C}$ is the subset of $\{0, 1\}^r$ given by setcore($\mathcal{C}$) $= \{p_C : C \in \text{core}(\mathcal{C})\}$. (The setcore is defined up to relabeling and twisting coordinates, since our definition would change if $U_1, \ldots, U_r$ were relabeled, or if the roles of U_i, V_i were swapped.)

Given a clutter $\mathcal{C}$ over ground set V, we may obtain a new clutter by *duplicating* a chosen element $v \in V$; specifically, we obtain the clutter $\mathcal{C}'$ over ground set $V \cup \{v'\}$ where $v' \notin V$, given by

$$\mathcal{C}' = \{C \cup \{v'\} : C \in \mathcal{C}, v \in C\} \cup \{C : C \in \mathcal{C}, v \notin C\}.$$

In general, if $\mathcal{C}'$ can be obtained from $\mathcal{C}$ by a finite number of duplications, we say that $\mathcal{C}'$ is a *duplication* of $\mathcal{C}$. Note that:

Remark 5.3.9. *Given a clean tangled clutter $\mathcal{C}$,* core($\mathcal{C}$) *is a duplication of the cuboid of* setcore($\mathcal{C}$).

Now consider the task of finding a fractional packing of $\mathcal{C}$ of value two. By definition, our fractional packing may only use members in core($\mathcal{C}$), and by Lemma 5.3.6, our fractional packing must assign a total weight of 1 to members C with $(p_C)_i = 0$, and a total weight of 1 to members C with $(p_C)_i = 1$, for each $i \in [r]$. Therefore, after a $\frac{1}{2}$-scaling, finding a fractional packing of value two becomes equivalent to expressing $\frac{1}{2} \cdot \mathbf{1} \in [0,1]^r$ as a convex combination of setcore($\mathcal{C}$). Then Theorem 5.3.4 implies:

Remark 5.3.10. *If $\mathcal{C}$ is a clean tangled clutter, then the convex hull of* setcore($\mathcal{C}$) *contains $\frac{1}{2} \cdot \mathbf{1}$. Moreover, for each $x \in$* setcore($\mathcal{C}$)*, we can write $\frac{1}{2} \cdot \mathbf{1}$ as a convex combination of* setcore($\mathcal{C}$) *which assigns a nonzero weight to x.*

Ideal minimally non-packing clutters of rank three

We just saw in Remark 5.3.10 that if $\mathcal{C}$ is a clean tangled clutter, then the setcore contains the center of the unit hypercube in its convex hull. If we additionally assume $\mathcal{C}$ does not pack and has rank three, i.e. $G(\mathcal{C})$ has exactly three connected components, then the setcore is determined up to twisting:

Lemma 5.3.11. *Let $\mathcal{C}$ be a clean tangled clutter with rank three, and assume that $(0,0,0) \in$* setcore($\mathcal{C}$) *and $\mathcal{C}$ does not pack. Then* setcore($\mathcal{C}$) $= \{(0,0,0),(1,1,0),(1,0,1),(0,1,1)\}$.

For the proof of the lemma, note that finding an integral packing of value two in $\mathcal{C}$ is equivalent to finding two points $p, q \in$ setcore($\mathcal{C}$) such that $p + q = \mathbf{1}$. That is, finding two disjoint members in $\mathcal{C}$ amounts to finding a pair of antipodal points in setcore($\mathcal{C}$).

Proof. Since $\mathcal{C}$ is tangled, we have $\tau(\mathcal{C}) = 2$. Since $\mathcal{C}$ does not pack, setcore($\mathcal{C}$) does not have antipodal points, so $(1,1,1) \notin$ setcore($\mathcal{C}$). By Remark 5.3.10, we can write $\left(\frac{1}{2},\frac{1}{2},\frac{1}{2}\right)$ as a convex combination of setcore($\mathcal{C}$) which assigns a nonzero weight to $(0,0,0)$. Since $\left(\frac{1}{2},\frac{1}{2},\frac{1}{2}\right)$ lies on the plane $x_1 + x_2 = 1$, and $(0,0,0)$ lies on one side of the plane, setcore($\mathcal{C}$) must contain a point on the other side, which must be $(1,1,0)$. Similarly, setcore($\mathcal{C}$) contains $(1,0,1)$ and $(0,1,1)$, and cannot contain any other points by the antipodality restriction. □

The hypothesis that setcore(C) $= \{(0,0,0),(1,1,0),(1,0,1),(0,1,1)\}$ guarantees the presence of four specific members of $\mathcal{C}$, which in turn imposes restrictions on the minimal covers of $\mathcal{C}$:

Lemma 5.3.12. *Let $\mathcal{C}$ be a clean tangled clutter with rank three, and denote by $\{U_1, V_1\}, \{U_2, V_2\}, \{U_3, V_3\}$ the bipartitions of the connected components of $G(\mathcal{C})$. Assume that*

$$C_1 = V_1 \cup V_2 \cup V_3 \qquad C_2 = V_1 \cup U_2 \cup U_3 \qquad C_3 = U_1 \cup V_2 \cup U_3 \qquad C_4 = U_1 \cup U_2 \cup V_3$$

are members of $\mathcal{C}$. Then:

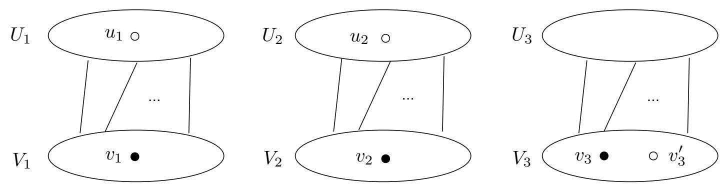

Figure 5.2: The minimal covers $\{v_1, v_2, v_3\}, \{u_1, u_2, v_3'\}$ in Lemma 5.3.12 (ii).

(i) *If C_i is a cover for some $i \in [4]$, then C_i contains a minimal cover of cardinality three, consisting of one element from each connected component of $G(\mathcal{C})$.*

(ii) *Assume that $\{v_1, v_2, v_3\}$ and $\{u_1, u_2, v_3'\}$ are minimal covers of $\mathcal{C}$ for some $u_1 \in U_1$, $u_2 \in U_2$, $v_1 \in V_1$, $v_2 \in V_2$, $v_3, v_3' \in V_3$ (see Figure 5.2). Then there exists a minimal cover B of $\mathcal{C}$ with $B \cap (U_1 \cup V_1) = \{u_1, v_1\}$ and $|B| \leq 3$.*

Proof. **(i)** By symmetry, it suffices to prove the statement for C_1. Choose a cover $B \subseteq V_1 \cup V_2 \cup V_3$ of minimum cardinality. Then B intersects each V_i, else B would be disjoint from one of C_2, C_3, C_4. Suppose for contradiction $|B \cap V_1| \geq 2$, and consider $\mathcal{C}' = \mathcal{C} \setminus I/J$ for $I = B - V_1$, $J = (U_2 \cup V_2 \cup U_3 \cup V_3) - B$, so that $\mathcal{C}'$ has ground set $U_1 \cup V_1$. We have two cases:

Case 1: $\tau(\mathcal{C}') \geq 2$. Since $\mathcal{C}'$ is a minor of $\mathcal{C}$, it follows that $\mathcal{C}'$ is clean. Since the induced subgraph $G(\mathcal{C})[U_1 \cup V_1]$ is a subgraph of $G(\mathcal{C}')$, it follows that $\mathcal{C}'$ is tangled. Then $G(\mathcal{C}')$ is bipartite by Theorem 5.3.3 (i). The subgraph relation implies that $G(\mathcal{C}')$ is connected and has bipartition $\{U_1, V_1\}$, so U_1 is a member of $\mathcal{C}'$ by Theorem 5.3.3 (iii). But $B - I = B \cap V_1$ is a cover of $\mathcal{C}'$ disjoint from U_1, a contradiction.

Case 2: $\tau(\mathcal{C}') \leq 1$; that is, there exists $D \in b(\mathcal{C})$ with $D \subseteq U_1 \cup V_1 \cup I$ and $|D \cap (U_1 \cup V_1)| \leq 1$. Therefore, if D contains an element of U_1, then $D \cap V_1 = \emptyset$, so D is disjoint from $C_4 = V_1 \cup U_2 \cup U_3$, a contradiction. Therefore, D is disjoint from U_1, so $D \subseteq V_1 \cup V_2 \cup V_3$, and

$$|D| \leq |D - I| + |I| \leq 1 + |B - V_1| < |B \cap V_1| + |B - V_1| = |B|,$$

a contradiction to the minimality of B. Hence $|B \cap V_1| = 1$, and similarly $|B \cap V_2| = 1$ and $|B \cap V_3| = 1$.

(ii) Suppose otherwise. Let $J = (U_1 \cup V_1) - \{u_1, v_1\}$, and let $\mathcal{C}' = \mathcal{C} \setminus \{u_1, v_1\}/J$; then our contrary assumption implies $\tau(\mathcal{C}') \geq 2$. Since $\mathcal{C}'$ is a minor of $\mathcal{C}$, it follows that $\mathcal{C}'$ is clean. Since the induced subgraph $G(\mathcal{C})[U_2 \cup V_2 \cup U_3 \cup V_3]$ is a subgraph of $G(\mathcal{C}')$, it follows that $\mathcal{C}'$ is tangled. Then $G(\mathcal{C}')$ is bipartite by Theorem 5.3.3 (i).

Note that u_2, v_2 are connected by an odd length path in $G(\mathcal{C})[U_2 \cup V_2]$ and hence also in $G(\mathcal{C}')[U_2 \cup V_2]$. Similarly, v_3, v_3' are connected by an even length path in $G(\mathcal{C})[U_3 \cup V_3]$ and hence also in $G(\mathcal{C}')[U_3 \cup V_3]$. But since $\{v_1, v_2, v_3\}$ and

$\{u_1, u_2, v_3'\}$ are minimal covers of $\mathcal{C}$, it follows that $\{u_2, v_3'\}, \{v_2, v_3\}$ are covers of $\mathcal{C}'$, hence edges of $G(\mathcal{C}')$, which gives an odd cycle in $G(\mathcal{C}')$, a contradiction. □

The following theorem summarizes the discussion in this section:

Theorem 5.3.13. *Let $\mathcal{C}$ be an ideal minimally non-packing clutter with covering number two and rank three. Then:*

(i) $\mathcal{C}$ is a clean tangled clutter.

(ii) $G(\mathcal{C})$ is bipartite, and the bipartitions of the connected components may be labeled as $\{U_1, V_1\}, \{U_2, V_2\}, \{U_3, V_3\}$, in such a way that $\text{setcore}(\mathcal{C}) = \{(0,0,0), (1,1,0), (1,0,1), (0,1,1)\}$*, so that the following are members of $\mathcal{C}$:*

$$\begin{aligned} C_1 &= V_1 \cup V_2 \cup V_3, \\ C_2 &= U_1 \cup U_2 \cup V_3, \\ C_3 &= U_1 \cup V_2 \cup U_3, \\ C_4 &= V_1 \cup U_2 \cup U_3 \end{aligned}$$

(iii) Each of C_1, C_2, C_3, C_4 contains a minimal cover of cardinality three, consisting of one element from each connected component of $G(\mathcal{C})$.

(iv) If $\{v_1, v_2, v_3\}$ and $\{u_1, u_2, v_3'\}$ are minimal covers of $\mathcal{C}$ for some $u_1 \in U_1$, $u_2 \in U_2$, $v_1 \in V_1$, $v_2 \in V_2$, $v_3, v_3' \in V_3$, then there exists a minimal cover B of $\mathcal{C}$ with $B \cap (U_1 \cup V_1) = \{u_1, v_1\}$ and $|B| \leq 3$, and similarly, there exists a minimal cover B' of $\mathcal{C}$ with $B' \cap (U_2 \cup V_2) = \{u_2, v_2\}$ and $|B'| \leq 3$. (The analogous statements obtained by using any of U_1, U_2, U_3, V_1, V_2 in the place of V_3 also hold.)

Proof. Statement (i) follows from Remark 5.3.2. By Theorem 5.3.3 (i), $G(\mathcal{C})$ is bipartite, and by Theorem 5.3.4, core($\mathcal{C}$) is nonempty, so we may label the bipartitions of the connected components as $\{U_1, V_1\}, \{U_2, V_2\}, \{U_3, V_3\}$ in such a way that $(0,0,0) \in$ setcore($\mathcal{C}$). Then by Lemma 5.3.11, we have $\text{setcore}(\mathcal{C}) = \{(0,0,0), (1,1,0), (1,0,1), (0,1,1)\}$, so that C_1, C_2, C_3, C_4 are members of $\mathcal{C}$.

Now we observe that C_1 is also a cover of $\mathcal{C}$, since otherwise, $\mathcal{C}$ would have a member disjoint from C_1, resulting in a pair of disjoint members, contradicting the assumption that $\mathcal{C}$ does not pack. Similarly, C_2, C_3, C_4 are covers of $\mathcal{C}$. Hence (iii) and (iv) follow from Lemma 5.3.12. □

Consider the new ideal minimally non-packing clutters of Theorem 5.1.2. We leave it to the reader to verify that the four minimal covers $\{2,4,6\}$, $\{2,3,5\}$, $\{1,4,5'\}$, $\{1,3,6'\}$ "come from" Theorem 5.3.13 (iii), while the two minimal covers $\{3,5,6'\}, \{4,5',6\}$ come from Theorem 5.3.13 (iv).

The well-known clutter Q_6 over ground set $[6]$ is defined by

$$Q_6 = \{\{2,4,6\}, \{1,3,6\}, \{1,4,5\}, \{2,3,5\}\}.$$

Theorem 5.3.13 (ii) and Remark 5.3.9 immediately imply:

Corollary 5.3.14. *Let $\mathcal{C}$ be an ideal minimally non-packing clutter with covering number two and rank three. Then* $\text{core}(\mathcal{C})$ *is a duplication of* Q_6.

It is known that all ideal minimally non-packing clutters with covering number two share a weaker property:

Theorem 5.3.15 ([8])**.** *Let $\mathcal{C}$ be an ideal minimally non-packing clutter with covering number two. Then some subset of $\mathcal{C}$ is a duplication of* Q_6.

In general, this Q_6-like subset need not be $\text{core}(\mathcal{C})$ itself, since $\text{rank}(\mathcal{C})$ may be greater than three. It is a feature of our construction that the clutters obtained, having rank three, exhibit Q_6-like structure in their core.

5.4 Previously known constructions

We conclude by describing the small instances of ideal minimally non-packing clutters provided by Cornuéjols, Guenin and Margot [8]. These constructions give thirteen ideal minimally non-packing clutters with covering number two and rank three. In Figures 5.3 and 5.4, these clutters are depicted via their blockers; cardinality-two minimal covers are shown as edges of a bipartite graph, and cardinality-three minimal covers are listed below the graph (and there are no other minimal covers).

The first twelve clutters (see Figure 5.3) are obtained by a common construction and are denoted $Q_6 \otimes X$, where $X \subseteq [6]$, subject to some restrictions on X. The construction produces clutters with exactly four minimal covers of cardinality three and no minimal covers of higher cardinality. The clutter $Q_6 \otimes \emptyset$ is just Q_6. The thirteenth clutter (see Figure 5.4) is a one-off example not conforming to the construction, and has the following incidence matrix:

$$\begin{array}{cccccccc} 1 & 1' & 2 & 2' & 3 & 4 & 5 & 6 \end{array}$$
$$\begin{pmatrix} 1 & 1 & 0 & 0 & 1 & 0 & 1 & 0 \\ 1 & 1 & 0 & 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 & 1 & 0 & 0 & 1 \\ 0 & 0 & 1 & 1 & 0 & 1 & 1 & 0 \\ 1 & 0 & 1 & 1 & 1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 & 0 & 1 & 0 & 1 \\ 0 & 1 & 1 & 0 & 1 & 0 & 0 & 1 \\ 1 & 1 & 0 & 1 & 0 & 1 & 1 & 0 \end{pmatrix}$$

All of these clutters share the properties required by Theorem 5.3.13. For instance, for the thirteenth clutter with the incidence matrix shown above, we have $U_1 = \{1, 1'\}, V_1 = \{2, 2'\}$, $U_2 = \{3\}, V_2 = \{4\}$ and $U_3 = \{6\}, V_3 = \{5\}$. The first four rows form the core, reaffirming (ii). The four members contain the minimal covers $\{1', 3, 5\}, \{1, 4, 6\}, \{2', 3, 6\}, \{2, 4, 5\}$, respectively, reaffirming (iii). Lastly, the five cardinality-two minimal covers, along with the minimal cover $\{1, 2', 6\}$, reaffirm (iv).

Late note on dijoins. After writing this paper, we noticed an intimate connection between our findings and some results on *dijoins* from the early 2000s. Let us explain this connection.

Ideal minimally non-packing clutters arise naturally from directed graphs. Let $D = (V, A)$ be a digraph. A *dicut* is a cut of the form $\delta^+(U) \subseteq A$ where $\delta^-(U) = \emptyset$, for some nonempty and proper subset U of V. A *dijoin* is any cover of the clutter of minimal dicuts. Equivalently, $J \subseteq A$ is a dijoin if D/J is strongly connected.

Denote by $\mathcal{C}(D)$ be the clutter of minimal dijoins of D. It follows from a well-known result of Lucchesi and Younger that $\mathcal{C}(D)$ is ideal [16]. *Woodall's Conjecture* states that the clutter $\mathcal{C}(D)$ must pack, a problem that has remained open to this date despite its simple statement [22]. The history of the problem is further muddled by the fact that $\mathcal{C}$ does *not* have the packing property. (Deletion in the clutter does *not* correspond to deletion in the digraph.)

Consider the three digraphs D_1, D_2, D_3 depicted in Figures 5.5, 5.6 and 5.7. The first digraph is due to Schrijver [17], while the other two are due to Cornuéjols and Guenin [7]. These digraphs were found as counterexamples to a conjecture of Edmonds and Giles on dijoins [11]. Even though $\mathcal{C}(D_i), i \in [3]$ pack, they do not have the packing property. More specifically, for $i \in [3]$, denote by I_i the arc subset corresponding to the dashed arcs of D_i. Then the minor $\mathcal{C}(D_i) \setminus I_i, i \in [3]$ is an ideal minimally non-packing clutter with covering number two. In fact, $\mathcal{C}(D_1) \setminus I_1 = Q_6 \otimes \{2, 4, 5\}$. The blocker of $\mathcal{C}(D_i) \setminus I_i$ is depicted next to D_i, where cardinality-two minimal covers are shown as edges of a bipartite graph, and all the other minimal covers are listed below the graph. The reader will notice that each $\mathcal{C}(D_i) \setminus I_i, i \in [3]$ has rank three. This can already be observed by looking at the digraph, as the solid arcs form three connected components, each of which is an alternating path of sources and sinks.

For more information, we refer the reader to Aaron Williams's very interesting Master's thesis [21]. He has shown that, up to a novel reduction called *folding*, the three clutters $\mathcal{C}(D_i) \setminus I_i, i \in [3]$ are the only ideal minimally non-packing clutters of covering number two and rank three coming from dijoins (see Chapter 6). He has also shown that $\mathcal{C}(D) \setminus I$ cannot be ideal minimally non-packing with covering number two and rank four (see Chapter 7).

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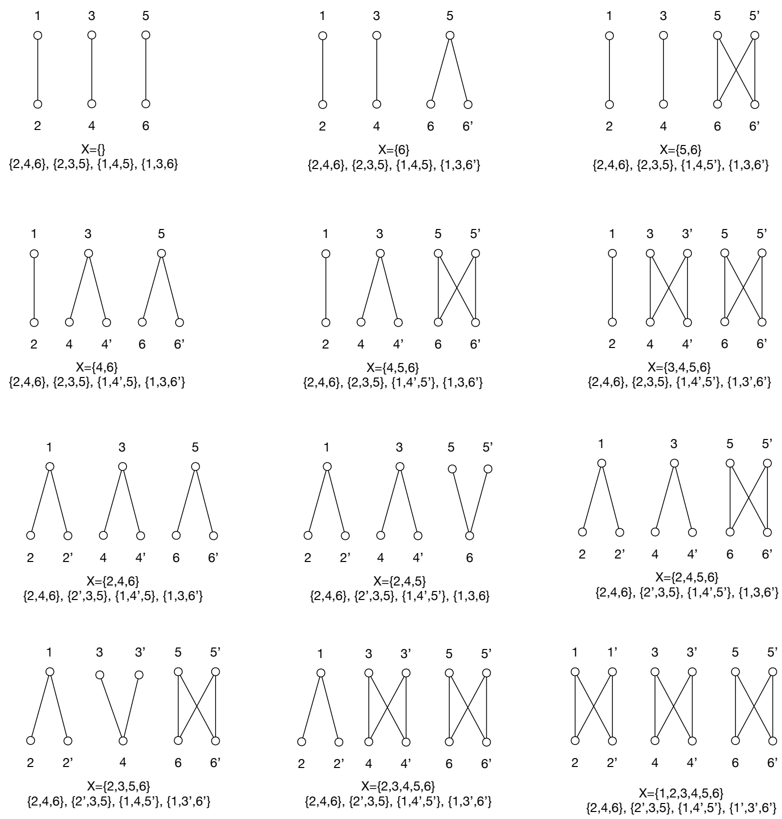
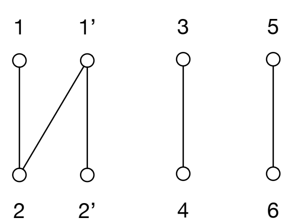


Figure 5.3: The blockers of the twelve clutters $Q_6 \otimes X$ from [8].



{1',3,5}, {1,4,6}, {2,4,5}, {2',3,6} and {1,2',6}

Figure 5.4: The blocker of the thirteenth ideal minimally non-packing clutter of rank three from [8].

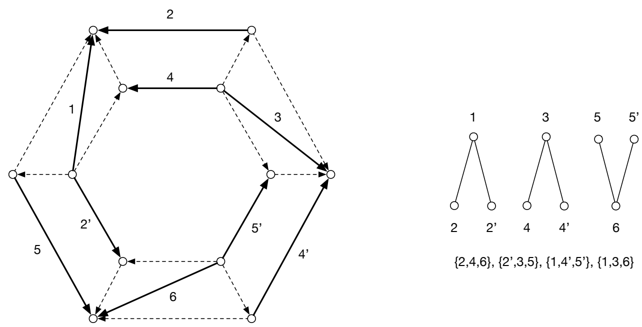


Figure 5.5: Left: The digraph D_1 from [17]. The unlabeled dashed arcs form the set I_1. Right: An illustration of the blocker of $\mathcal{C}(D_1) \setminus I_1$.

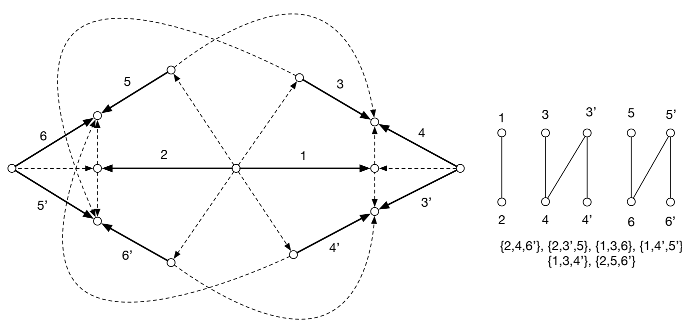


Figure 5.6: Left: The digraph D_2 from [7]. The unlabeled dashed arcs form the set I_2. Right: An illustration of the blocker of $\mathcal{C}(D_2) \setminus I_2$.

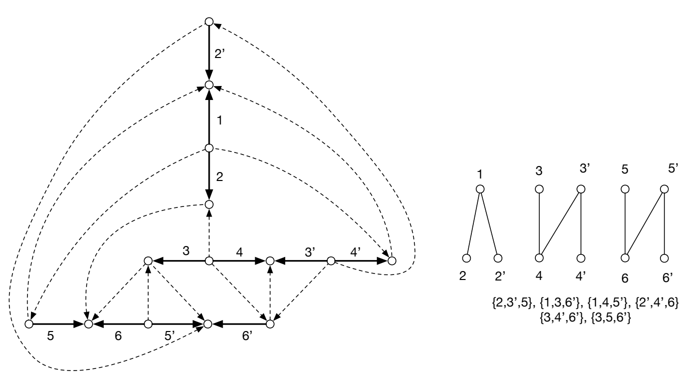


Figure 5.7: Left: The digraph D_3 from [7]. The unlabeled dashed arcs form the set I_3. Right: An illustration of the blocker of $\mathcal{C}(D_3) \setminus I_3$.

References

[1] Ahmad Abdi, Gérard Cornuéjols, and Dabeen Lee. "Intersecting restrictions in clutters". In: *Combinatorica* 40.5 (2020), pp. 605–623.

[2] Ahmad Abdi and Dabeen Lee. "Deltas, extended odd holes and their blockers". In: *J. Combin. Theory Ser. B* 136 (2019), pp. 193–203.

[3] Ahmad Abdi, Kanstantsin Pashkovich, and Gérard Cornuéjols. "Ideal clutters that do not pack". In: *Math. Oper. Res.* 43.2 (2018), pp. 533–553.

[4] Ahmad Abdi et al. "Cuboids, a class of clutters". In: *J. Combin. Theory Ser. B* 142 (2020), pp. 144–209.

[5] Ahmad Abdi et al. "Idealness of k-wise intersecting families". In: *Integer programming and combinatorial optimization*. Vol. 12125. Lecture Notes in Comput. Sci. Springer, Cham, 2020, pp. 1–12.

[6] Michele Conforti and Gérard Cornuéjols. *Clutters that pack and the max flow min cut property: a conjecture*. Tech. rep. The Fourth Bellairs Workshop on Combinatorial Optimization, 1993.

[7] Gérard Cornuéjols and Bertrand Guenin. "On dijoins". In: *Discrete Math.* 243.1-3 (2002), pp. 213–216.

[8] Gérard Cornuéjols, Bertrand Guenin, and François Margot. "The packing property". In: *Math. Program.* 89.1, Ser. A (2000), pp. 113–126.

[9] Gérard Cornuéjols and Beth Novick. "Ideal $0, 1$ matrices". In: *J. Combin. Theory Ser. B* 60.1 (1994), pp. 145–157.

[10] Jack Edmonds and D. R. Fulkerson. "Bottleneck extrema". In: *J. Combinatorial Theory* 8 (1970), pp. 299–306.

[11] Jack Edmonds and Rick Giles. "A min-max relation for submodular functions on graphs". In: *Studies in integer programming (Proc. Workshop, Bonn, 1975)*. 1977, 185–204. Ann. of Discrete Math., Vol. 1.

[12] D. R. Fulkerson. "Blocking and anti-blocking pairs of polyhedra". In: *Math. Programming* 1 (1971), pp. 168–194.

[13] J. R. Isbell. "A class of simple games". In: *Duke Math. J.* 25 (1958), pp. 423–439.

[14] Alfred Lehman. "On the width-length inequality". In: *Math. Programming* 16.2 (1979), pp. 245–259.

[15] Alfred Lehman. "The width-length inequality and degenerate projective planes". In: *Polyhedral combinatorics (Morristown, NJ, 1989)*. Vol. 1. DIMACS Ser. Discrete Math. Theoret. Comput. Sci. Amer. Math. Soc., Providence, RI, 1990, pp. 101–105.

[16] C. L. Lucchesi and D. H. Younger. "A minimax theorem for directed graphs". In: *J. London Math. Soc. (2)* 17.3 (1978), pp. 369–374.

[17] A. Schrijver. "A counterexample to a conjecture of Edmonds and Giles". In: *Discrete Math.* 32.2 (1980), pp. 213–215.

[18] P. D. Seymour. "On Lehman's width-length characterization". In: *Polyhedral combinatorics (Morristown, NJ, 1989)*. Vol. 1. DIMACS Ser. Discrete Math. Theoret. Comput. Sci. Amer. Math. Soc., Providence, RI, 1990, pp. 107–117.

[19] P. D. Seymour. "The forbidden minors of binary clutters". In: *J. London Math. Soc. (2)* 12.3 (1975/76), pp. 356–360.

[20] P. D. Seymour. "The matroids with the max-flow min-cut property". In: *J. Combinatorial Theory Ser. B* 23.2-3 (1977), pp. 189–222.

[21] Aaron Williams. "Packing directed joins". MA thesis. University of Waterloo, 2004.

[22] D. R. Woodall. "Menger and König systems". In: *Theory and applications of graphs (Proc. Internat. Conf., Western Mich. Univ., Kalamazoo, Mich., 1976)*. Vol. 642. Lecture Notes in Math. Springer, Berlin, 1978, pp. 620–635.

Chapter 6

A nonlinear Lazarev-Lieb theorem: L^2-orthogonality via motion planning

Joint work with Florian Frick.

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6.1 Introduction

In 1965 Hobby and Rice established the following result:

Theorem 6.1.1 (Hobby and Rice [4]). *Let* $f_1, \dots, f_n \in L^1([0,1];\mathbb{R})$. *Then there exists* $h\colon [0,1] \to \{\pm 1\}$ *with at most* n *sign changes, such that for all* j,

$$\int_0^1 f_j(x)h(x)dx = 0.$$

If we restrict the f_j to lie in $L^2([0,1];\mathbb{R})$, we can view this as an orthogonality result in the L^2 inner product. The Hobby–Rice theorem and its generalizations have found a multitude of applications, ranging from mathematical physics [6] and combinatorics [1] to the geometry of spatial curves [2].

The theorem also holds for $f_1, \dots, f_n \in L^1([0,1];\mathbb{C})$, provided h is allowed $2n$ sign changes, by splitting the f_j into real and imaginary parts. Lazarev and Lieb showed that for complex-valued f_j, the function h can be chosen in $C^\infty([0,1];S^1)$, where S^1 denotes the unit circle in $\mathbb{C}$:

Theorem 6.1.2 (Lazarev and Lieb [5]). *Let $f_1, \dots, f_n \in L^1([0,1];\mathbb{C})$. Then there exists $h \in C^\infty([0,1];S^1)$ such that for all j,*

$$\int_0^1 f_j(x)h(x)dx = 0.$$

If h is obtained by smoothing the function h_0 guaranteed by Theorem 6.1.1, then we would expect its $W^{1,1}$-norm, given by

$$\|h\|_{W^{1,1}} = \int_0^1 |h(x)|dx + \int_0^1 |h'(x)|dx$$

to be approximately $1 + 2\pi n$, since $|h(x)| = 1$, and each sign change of h_0 contributes approximately π to $\int_0^1 |h'(x)|dx$. However, Lazarev and Lieb did not establish any bound on the $W^{1,1}$-norm of h and left this as an open problem; this was accomplished by Rutherfoord [9], who established a bound of $1 + 5\pi n$. Here we improve this bound to $1 + 2\pi n$; see Corollary 6.1.4.

The Hobby–Rice theorem has a simple proof due to Pinkus [8] via the Borsuk–Ulam theorem, which states that any map $f\colon S^n \to \mathbb{R}^n$ with $f(-x) = -f(x)$ for all $x \in S^n$ has a zero. Lazarev and Lieb asked whether there is a similar proof of their result and write: "There seems to be no way to adapt the proof of the Hobby–Rice Theorem (which involves a fixed-point argument)." Rutherfoord [9] offered a simplified proof of Theorem 6.1.2 based on Brouwer's fixed point theorem. Here we give a proof using the Borsuk–Ulam theorem directly, which adapts Pinkus' proof of the Hobby–Rice theorem. The advantage of this approach is that our main result gives a nonlinear extension of the result of Lazarev and Lieb; see Section 6.4 for the proof:

Theorem 6.1.3. *Let $\psi\colon C^\infty([0,1];S^1) \to \mathbb{R}^n$ be continuous with respect to the L^1-norm such that $\psi(-h) = -\psi(h)$ for all $h \in C^\infty([0,1];S^1)$. Then there exists $h \in C^\infty([0,1];S^1)$ with $\psi(h) = 0$ and $\|h\|_{W^{1,1}} \le 1 + \pi n$.*

This is a non-linear extension of Theorem 6.1.2 since for given $f_1, \dots, f_n \in L^1([0,1];\mathbb{C})$ the map $\psi(h) = (\int_0^1 f_j(x)h(x)dx)_j$ is continuous (see Section 6.2) and linear, so in particular, ψ satisfies $\psi(-h) = -\psi(h)$. Using the L^1-norm is no restriction; as we show in the next section, the L^p norms on $C^\infty([0,1];S^1)$ for $1 \le p < \infty$ are all equivalent, so we could replace L^1 with any such L^p. In fact, the only relevant feature of the L^1-norm is that functions h_1, h_2 are close in the L^1-norm if h_1, h_2 are uniformly close outside of a set of small measure. As a consequence, we recover the result of Lazarev and Lieb, with a $W^{1,1}$-norm bound of $1 + 2\pi n$ since ψ takes values in $\mathbb{C}^n \cong \mathbb{R}^{2n}$; see Section 6.2 for the proof:

Corollary 6.1.4. *Let $f_1, \dots, f_n \in L^1([0,1];\mathbb{C})$. Then there exists $h \in C^\infty([0,1];S^1)$ with $\|h\|_{W^{1,1}} \le 1 + 2\pi n$ such that for all j,*

$$\int_0^1 f_j(x)h(x)dx = 0.$$

Given a space Z with a $\mathbb{Z}/2$-action $\sigma\colon Z \to Z$, the largest integer n such that the n-sphere S^n with the antipodal $\mathbb{Z}/2$-action (i.e. $x \mapsto -x$) admits a continuous map $f\colon S^n \to Z$ with $f(-x) = \sigma(f(x))$ for all $x \in S^n$ is called the $\mathbb{Z}/2$-coindex of Z, denoted coind Z. We show that the coindex of the space of smooth S^1-valued functions in the L^1-norm with $W^{1,1}$-norm at most $1 + \pi n$ is between n and $2n - 1$; see Theorem 6.6.2. Determining the coindex exactly remains an interesting open problem. Our proof proceeds by constructing $\mathbb{Z}/2$-maps from S^n, i.e., commuting with the antipodal $\mathbb{Z}/2$-actions, via elementary obstruction theory, that is, inductively dimension by dimension.

We find it illuminating to phrase our proof using the language of motion planning algorithms. A motion planning algorithm (mpa) for a space Z is a continuous choice of connecting path for any two endpoints in Z; see Section 6.3 for details and Farber [3] for an introduction. An mpa for Z exists if and only if Z is contractible. Here we introduce the notion of (full) lifted mpa, which does not imply contractibility but is sufficiently strong to establish lower bounds for the coindex of Z; see Theorem 6.3.5. We refer to Section 6.3 for details.

6.2 Relationship between topologies on $C^\infty([0,1];S^1)$

We now make precise our introductory comments about the topologies on $C^\infty([0,1];S^1)$ induced by the various L^p-norms. We claimed that “the only relevant feature of the L^1-norm is that functions h_1, h_2 are close in the L^1-norm if h_1, h_2 are uniformly close outside of a set of small measure.” To give content to this statement, we define a metric $d_{0,\infty}$ on $C^\infty([0,1];S^1)$ by

$$\begin{aligned} d_{0,\infty}(h_1,h_2) = \inf\{\delta > 0 : |h_2(x) - h_1(x)| < \delta \text{ for all } x \in [0,1] \setminus S, \\ \text{for some } S \subseteq [0,1] \text{ with } \mu(S) < \delta\}. \end{aligned}$$

Proposition 6.2.1. *The function $d_{0,\infty}$ is a metric.*

Proof. By the continuity of maps in $C^\infty([0,1];S^1)$, we have $d_{0,\infty}(h_1,h_2) = 0$ iff $h_1 = h_2$. For the triangle inequality, suppose:

- $|h_2(x) - h_1(x)| < \delta_1$ for all $x \in [0,1] \setminus S_1$, where $\mu(S_1) < \delta_1$.
- $|h_3(x) - h_2(x)| < \delta_2$ for all $x \in [0,1] \setminus S_2$, where $\mu(S_2) < \delta_2$

Then $|h_3(x) - h_1(x)| < \delta_1 + \delta_2$ for all $x \in [0,1] \setminus (S_1 \cup S_2)$, and $\mu(S_1 \cup S_2) < \delta_1 + \delta_2$. Hence $d_{0,\infty}(h_1,h_3) \le \delta_1 + \delta_2$. Taking the infimum over δ_1, δ_2, we obtain $d_{0,\infty}(h_1,h_3) \le d_{0,\infty}(h_1,h_2) + d_{0,\infty}(h_2,h_3)$. □

We consider $C^\infty([0,1];S^1)$ under topologies induced by various norms or metrics:

Definition 6.2.2. Define spaces $Z_p, Z_{p,\mu}, Z_{0,\infty}$ as $C^\infty([0,1];S^1)$, equipped with the topology induced by the norm or metric indicated in the following table. (Here μ is a measure on $[0,1]$; the notation $\mu \ll \lambda$ indicates that μ is absolutely continuous with respect to Lebesgue measure.)

Name of space	Restrictions	Norm or metric
Z_p	$1 \le p \le \infty$	L^p-norm
$Z_{p,\mu}$	$1 \le p < \infty$, $\mu \ll \lambda$, μ finite	L^p-norm w.r.t. μ
$Z_{0,\infty}$	(none)	$d_{0,\infty}$ metric

Proposition 6.2.3. *The L^p-norms for $1 \le p < \infty$ induce equivalent topologies on $C^\infty([0,1]; S^1)$.*

Proof. Note that $\|h\|_p < \infty$ for all $h \in C^\infty([0,1]; S^1)$, so the identity maps $1_{p,q} \colon Z_p \to Z_q$ are well-defined as functions. It suffices to show that $1_{p,q}$ is continuous for all $p, q \in [1, \infty)$.

It is a standard fact that $1_{p,q}$ is continuous for $p \ge q$ when the domain has finite measure, as is the case here for $[0,1]$. For $p < q$, we have

$$\begin{aligned}\|h_2 - h_1\|_q &= \left(\int_0^1 |h_2(x) - h_1(x)|^q dx\right)^{1/q} \\ &\le \left(\int_0^1 |h_2(x) - h_1(x)|^p \cdot (\text{diam}(S^1))^{q-p} dx\right)^{1/q} \\ &\le (\text{diam}(S^1))^{(q-p)/q} \cdot \|h_2 - h_1\|_p^{p/q}\end{aligned}$$

Since S^1 is bounded, $1_{p,q}$ is continuous. Hence the Z_p are all homeomorphic. □

Proposition 6.2.4. *The metric $d_{0,\infty}$ and the L^1-norm induce equivalent topologies on $C^\infty([0,1]; S^1)$.*

Proof. It suffices to show that the identity maps between $Z_{0,\infty}, Z_1$ are continuous.

For the identity map $1 \colon Z_{0,\infty} \to Z_1$, suppose $d_{0,\infty}(h_1, h_2) < \delta$, so that there exists $S \subseteq [0,1]$ with $\mu(S) < \delta$ such that $|h_2(x) - h_1(x)| < \delta$ on $[0,1] \setminus S$. Then

$$\int_0^1 |h_2(x) - h_1(x)| dx \le \int_S \text{diam}(S^1) dx + \int_{[0,1]\setminus S} \delta dx \le \delta(\text{diam}(S^1) + 1).$$

This shows that $1 \colon Z_{0,\infty} \to Z_1$ is continuous.

For the identity map $1 \colon Z_1 \to Z_{0,\infty}$, let $\varepsilon > 0$ and suppose $\|h_2 - h_1\|_1 < \delta$ for $\delta = \varepsilon^2$. If $d_{0,\infty}(h_1, h_2) \ge \varepsilon$, then $|h_2(x) - h_1(x)| \ge \varepsilon$ on a set S with $\mu(S) \ge \varepsilon$, implying $\|h_2 - h_1\|_1 \ge \varepsilon^2$, a contradiction. Hence $d_{0,\infty}(h_1, h_2) < \varepsilon$, and $1 \colon Z_1 \to Z_{0,\infty}$ is continuous. □

We now show that if a finite measure μ is absolutely continuous with respect to Lebesgue measure, then μ can only produce coarser topologies than Lebesgue measure:

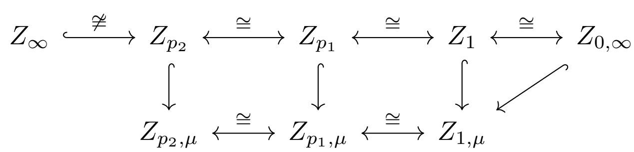
Proposition 6.2.5. *The identity function $1 \colon Z_1 \to Z_{1,\mu}$ is continuous.*

Proof. By Proposition 6.2.4, it suffices to show that $1\colon Z_{0,\infty} \to Z_{1,\mu}$ is continuous. The argument is similar to the argument that $1\colon Z_{0,\infty} \to Z_1$ is continuous. Using λ to denote Lebesgue measure, suppose $d_{0,\infty}(h_1, h_2) < \delta$, so that there exists $S \subseteq [0,1]$ with $\lambda(S) < \delta$ such that $|h_2(x) - h_1(x)| < \delta$ on $[0,1] \setminus S$. Then

$$\begin{aligned}\int_{[0,1]} |h_2(x) - h_1(x)| d\mu &\leq \int_S \text{diam}(S^1) d\mu + \int_{[0,1]\setminus S} \delta d\mu \\ &\leq \text{diam}(S^1)\mu(S) + \delta\mu([0,1])\end{aligned}$$

Note that since μ is finite, we have $\mu([0,1]) < \infty$. As $\delta \to 0$, we have $\lambda(S) \to 0$, so $\mu(S) \to 0$ by absolute continuity, hence the right side approaches 0. This shows the desired continuity. □

The relationships between the topologies on $C^\infty([0,1];S^1)$ can be summarized as follows, where the spaces are as defined in Definition 6.2.2, and $1 < p_1 < p_2 < \infty$:



Therefore, when establishing the continuity of ψ for the sake of applying Theorem 6.1.3, we may use any L^p norm on $C^\infty([0,1];S^1)$, with respect to any finite measure μ on $[0,1]$ which is absolutely continuous with respect to Lebesgue measure. (If we use a measure μ other than Lebesgue measure, we can precompose ψ with $1\colon Z_1 \to Z_{1,\mu}$ before applying Theorem 6.1.3.)

With these results in hand, we can now deduce Corollary 6.1.4 from Theorem 6.1.3:

Proof of Corollary 6.1.4. Let $\psi\colon C^\infty([0,1];S^1) \to \mathbb{C}^n$ be given by component maps

$$\psi_j\colon h \mapsto \int_0^1 f_j(x)h(x)dx.$$

We claim ψ_j is continuous. Since $f_j \in L^1([0,1];\mathbb{C})$, f_j induces a finite measure μ_f which is absolutely continuous with respect to Lebesgue measure, given by

$$\mu_f(S) = \int_0^1 |f_j(x)|dx.$$

By the above, we may view $C^\infty([0,1]; S^1)$ as having the topology induced by the L^1-norm $\|\cdot\|_1$ with respect to μ_f. Then

$$\begin{aligned}|\psi_j(h_2) - \psi_j(h_1)| &\leq \int_0^1 |f_j(x)| \cdot |h_2(x) - h_1(x)| dx \\ &\leq \int_{[0,1]} |h_2 - h_1| d\mu_f \\ &\leq \|h_2 - h_1\|_1.\end{aligned}$$

Therefore, ψ_j is continuous, so ψ is continuous. Viewing the codomain $\mathbb{C}^n$ of ψ as $\mathbb{R}^{2n}$, we may apply Theorem 6.1.3 and get $\|h\|_{W^{1,1}} \leq 1 + 2\pi n$. □

6.3 Lifts of motion planning algorithms and the coindex

Our proof of Theorem 6.1.3 makes use of *motion planning algorithms*; see Farber [3]. We use Y, Z in the following definitions to match our notation later:

Definition 6.3.1. Let Z be a topological space, and let PZ be the space of continuous paths $\gamma\colon [0,1] \to Z$, equipped with the compact-open topology. Then a *motion planning algorithm* (or *mpa*) is a continuous map $s\colon Z \times Z \to PZ$, such that $s(z_0, z_1)(0) = z_0$ and $s(z_0, z_1)(1) = z_1$.

For Z a locally compact Hausdorff space, using the compact-open topology for PZ ensures that a function $s\colon Z \times Z \to PZ$ is continuous if and only if its uncurried form $\widetilde{s}\colon Z \times Z \times [0,1] \to Z$ given by $(z_0, z_1, t) \mapsto s(z_0, z_1)(t)$ is continuous; see Munkres [7, Thm. 46.11]. One basic fact is that an mpa for Z exists if and only if Z is contractible [3].

We modify the definition above for our purposes:

Definition 6.3.2. Let Y, Z be topological spaces, and let $\phi\colon Y \to Z$ be continuous. Let $(\preceq)$ be a preorder (i.e., a reflexive, transitive binary relation) on Y, and let $Y^2_\preceq = \{(y_0, y_1) \in Y^2 : y_0 \preceq y_1\}$, giving Y^2 the product topology and $Y^2_\preceq$ the resulting subspace topology.

A *lifted motion planning algorithm* (or *lifted mpa*) for $(Y, Z, \phi, \preceq)$ is a family of maps $s_w\colon Y^2_\preceq \to PY$ for $w \in (0,1]$ with $s_w(y_0, y_1)(0) = y_0$ and $s_w(y_0, y_1)(1) = y_1$, assembling into a continuous map $s\colon (0,1] \times Y^2_\preceq \to PY$, with the following additional continuity property:

For all $y \in Y$ and all neighborhoods V of $\phi(y) \in Z$,
there exists a neighborhood U of $\phi(y) \in Z$ and $\delta > 0$ such that:
if $\phi(y_0), \phi(y_1) \in U$, $w < \delta$,
then $\phi(s_w(y_0, y_1)(t)) \in V$ for all $t \in [0,1]$.

Definition 6.3.3. A lifted mpa $s\colon (0,1] \times Y^2_\preceq \to PY$ for $(Y, Z, \phi, \preceq)$ is *full* if $y_0 \preceq y_1$ for all $y_0, y_1 \in Y$. In this case we say s is a full lifted mpa for (Y, Z, ϕ), omitting $(\preceq)$.

In our applications, the space Z will be a function space of interest (e.g. $C^\infty([0,1];S^1)$), where continuously assigning paths between functions is difficult. We will obtain Y from Z by replacing the codomain with a covering space (e.g., $C^\infty([0,1];\mathbb{R})$), so that continuously assigning paths between functions in Y is easier. We will satisfy the continuity requirement by making $s_w(g_0,g_1)(t)(x)$ equal $g_0(x)$ or $g_1(x)$ for all $x \in [0,1]$ outside of an interval of length w. (See Section 6.4 for details.)

In an mpa, paths between nearby points need not remain close to those points. However, if we are given an mpa $s\colon Z \times Z \to PZ$ with the property that the paths $s(z,z)$ are constant, we may construct a full lifted mpa for $(Z,Z,1_Z)$ by taking $s_w = s$ for all w; the continuity property just restates the continuity of s at diagonal points $(z,z) \in Z \times Z$.

Whereas the existence of an mpa implies contractibility, the existence of a lifted mpa yields lower bounds for the (equivariant) topology of Z. Recall that for a topological space Z with $\mathbb{Z}/2$-action generated by $\sigma\colon Z \to Z$ the $\mathbb{Z}/2$-coindex of Z denoted by $\operatorname{coind} Z$ is the largest integer n such that there is a $\mathbb{Z}/2$-map $f\colon S^n \to Z$, that is, a map satisfying $f(-x) = \sigma(f(x))$.

Definition 6.3.4. Let $x \in S^k$, and let $x = (x_1,\ldots,x_{k+1})$. We say that x is *positive* if its last nonzero coordinate is positive, and *negative* otherwise.

Our main tool in proving Theorem 6.1.3 will be the following theorem:

Theorem 6.3.5. *Let Y, Z be topological spaces, equip Y with a $\mathbb{Z}$-action generated by $\rho\colon Y \to Y$, and equip Z with a $\mathbb{Z}/2$-action generated by $\sigma\colon Z \to Z$. Let $\phi\colon Y \to Z$ be continuous and equivariant, i.e., $\sigma \circ \phi = \phi \circ \rho$. Let $(\preceq)$ be a preorder on Y and $s\colon (0,1] \times Y^2_{\preceq} \to PY$ a lifted mpa for $(Y,Z,\phi,\preceq)$ such that:*

(1) $y \preceq \rho(y)$.

(2) $\rho(y_0) \preceq \rho(y_1)$ *if and only if* $y_0 \preceq y_1$.

(3) $y_0 \preceq y_1$ *implies* $y_0 \preceq s_w(y_0,y_1)(t) \preceq y_1$*, for all* $w \in (0,1]$*,* $t \in [0,1]$.

Then for each integer $n \geq 0$, there exists a $\mathbb{Z}/2$-map $\beta_n\colon S^n \to Z$. Moreover, for any choice of initial point $y^* \in Y$, the maps β_n can be chosen such that β_n maps each positive point of S^n to a point in Z of the form $\phi(y)$, with $y^* \preceq y \preceq \rho^n(y^*)$, that is, the subspace of these points $\phi(y)$ and their antipodes $\sigma(\phi(y))$ in Z has coindex at least n.

We will later apply Theorem 6.3.5 by taking Z to be $C^\infty([0,1];S^1)$ and Y to be the space of increasing functions in $C^\infty([0,1];\mathbb{R})$, both with the L^1-norm. The last part of the theorem will give the desired $W^{1,1}$-norm bound.

Proof of Theorem 6.3.5. We will inductively construct a function $\alpha_n\colon S^n \to Y$ and then take $\beta_n = \phi \circ \alpha_n$. We will allow α_n to be discontinuous on the equator of S^n, but in such a way that $\phi \circ \alpha_n$ is continuous everywhere.

Specifically, let $\alpha_k\colon S^k \to Y$ be a function, not necessarily continuous. Let $m\colon S^k \to S^k$ be given by $(x_1, \ldots, x_k, x_{k+1}) \mapsto (x_1, \ldots, x_k, -x_{k+1})$, so that m mirrors points across the plane perpendicular to the last coordinate axis. Then we say that α_k is *good* if

(α-1) For x positive, $y^* \preceq \alpha_k(x) \preceq \rho^k(y^*)$, and $\alpha_k(-x) = \rho(\alpha_k(x))$.

(α-2) For x in the open upper hemisphere, $\alpha_k(x) \preceq \alpha_k(m(x))$.

(α-3) α_k is continuous on the open upper hemisphere.

(α-4) $\phi \circ \alpha_k$ is continuous.

Let $u, l\colon B^{k+1} \to S^k$ be the projections to the closed upper and lower hemispheres, that is, $u(x)$ is the unique point in the closed upper hemisphere sharing its first k coordinates with x, and similarly for $l(x)$ for the lower hemisphere. Then we have the following claim:

Claim 1. *If $\alpha_k\colon S^k \to Y$ is good, then α_k extends to $\widetilde{\alpha}_k\colon B^{k+1} \to Y$, such that:*

($\widetilde{\alpha}$-1) For all $x \in B^{k+1}$, we have $y^* \preceq \widetilde{\alpha}_k(x) \preceq \rho^{k+1}(writey^*)$.

($\widetilde{\alpha}$-2) For all $x \in B^{k+1}$, we have $\alpha_k(u(x)) \preceq \widetilde{\alpha}_k(x) \preceq \alpha_k(l(x))$.

($\widetilde{\alpha}$-3) $\widetilde{\alpha}_k$ is continuous in the interior of B^{k+1}.

($\widetilde{\alpha}$-4) $\phi \circ \widetilde{\alpha}_k$ is continuous.

Proof of Claim. Let $E \subset S^k$ be the equator, the set of points neither in the open upper or lower hemisphere. The set E is compact, so the distance $d(x, E)$ for $x \in B^{k+1}$ is well-defined and nonzero for $x \notin E$. Define $\widetilde{\alpha}_k\colon B^{k+1} \to X_{k+1}$ by

$$\widetilde{\alpha}_k(x) = \begin{cases} \alpha_k(x) & x \in E \\ s_{w(x)}(\alpha_k(u(x)), \alpha_k(l(x)))(t(x)) & x \notin E \end{cases}$$
$$\text{where } w(x) = \min(d(x, E), t(x), 1 - t(x))$$
$$t(x) = \frac{d(u(x), x)}{d(u(x), l(x))}$$

Note that $l(x) = m(u(x))$, so (α-2) implies $\alpha_k(u(x)) \preceq \alpha_k(l(x))$, so

$$s_{w(x)}(\alpha_k(u(x)), \alpha_k(l(x)))$$

is well-defined, and (3) gives $\alpha_k(u(x)) \preceq \widetilde{\alpha}(x) \preceq \alpha_k(l(x))$, establishing ($\widetilde{\alpha}$-2).

By (α-1), we have $\rho(y^*) \preceq \rho(\alpha_k(x)) \preceq \rho^{k+1}(y^*)$ for x negative, so $y^* \preceq \alpha_k(x) \preceq \rho^{k+1}(y^*)$ for all $x \in S^k$. Along with the inequality above, this implies $y^* \preceq \widetilde{\alpha}_k(x) \preceq \rho^{k+1}(y^*)$, establishing ($\widetilde{\alpha}$-1).

The function $\widetilde{\alpha}_k$ is continuous for $x \notin E$, since $u(-)$, $l(-)$, $d(-,-)$, $d(-, E)$ are all continuous, $u(x), l(x) \notin E$, and α_k is continuous on the open upper (and hence lower) hemisphere. In particular, $\widetilde{\alpha}_k$ is continuous in the interior of B^{k+1}, establishing ($\widetilde{\alpha}$-3).

It remains to show $\phi \circ \widetilde{\alpha}_k$ is continuous at $x \in E$. Let V be a neighborhood of $\phi(\widetilde{\alpha}_k(x)) = \phi(\alpha_k(x)) \in Z$, and obtain $\delta > 0$ and a neighborhood U of $\phi(\alpha_k(x)) \in Z$ as in the lifted mpa definition. Since $u(-), l(-), d(-, E)$ are continuous, there exists a neighborhood $W \subseteq B^{k+1}$ of x such that for all $x' \in W$ we have $d(x', E) < \delta$ and $u(x'), l(x') \in (\phi \circ \alpha_k)^{-1}(U)$, using the continuity of $\phi \circ \alpha_k$ given by (α-4). Then $\phi(\alpha_k(u(x'))), \phi(\alpha_k(l(x'))) \in U$, so the lifted mpa property implies $\phi(\widetilde{\alpha}_k(x)) \in V$, which shows $\phi \circ \widetilde{\alpha}_k$ is continuous at x, establishing ($\widetilde{\alpha}$-4). □

We use the claim above to inductively construct $\alpha_k \colon S^k \to Y$, by extending each α_k to a map $\widetilde{\alpha}_k \colon B^{k+1} \to Y$, using $\widetilde{\alpha}_k$ for the upper hemisphere of α_{k+1}, and extending to the negative hemisphere via $\alpha_{k+1}(-x) = \rho(\alpha_{k+1}(x))$. Specifically, we have the following claim:

Claim 2. *For all $k \geq 0$ there exists $\alpha_k \colon S^k \to Y$, not necessarily continuous, such that α_k is good.*

Proof of Claim. We use induction. For the base case, use ± 1 to denote the points of S^0; then let α_0 map ± 1 to $y^*, \rho(y^*)$, respectively. Then α_0 is good.

Given α_k good and $\widetilde{\alpha}_k$ obtained through the previous claim, we now construct $\alpha_{k+1} \colon S^{k+1} \to Y$. Let $\pi \colon S^{k+1}_{\geq 0} \to B^{k+1}$ be the projection of the closed upper hemisphere onto the first $k+1$ coordinates. We define maps on the two closed hemispheres as follows:

$$(\alpha_{k+1})_{\geq 0} \colon S^{k+1}_{\geq 0} \to Y \qquad x \mapsto \widetilde{\alpha}_k(\pi(x))$$
$$(\alpha_{k+1})_{\leq 0} \colon S^{k+1}_{\leq 0} \to Y \qquad x \mapsto \rho(\widetilde{\alpha}_k(\pi(-x)))$$

Finally, we define α_{k+1} by $x \mapsto (\alpha_{k+1})_{\geq 0}(x)$ for x positive and $x \mapsto (\alpha_{k+1})_{\leq 0}(x)$ for x negative.

For α_{k+1}, (α-1) holds by construction, due to ($\widetilde{\alpha}$-1). Next, since $\widetilde{\alpha}_k$ is continuous in the interior of B^{k+1}, we have that $(\alpha_{k+1})_{\geq 0}$ is continuous on the open upper hemisphere, hence α_{k+1} is also, so (α-3) holds also.

Since $\widetilde{\alpha}_k$ satisfies $\widetilde{\alpha}_k(-x) = \rho(\widetilde{\alpha}_k(x))$ for positive x on the boundary sphere $S^k \subset B^{k+1}$, we have $(\alpha_{k+1})_{\leq 0}(x) = \rho^2((\alpha_{k+1})_{\geq 0}(x))$ for positive x on the equator $S^k \subset S^{k+1}$, and $(\alpha_{k+1})_{\leq 0}(x) = (\alpha_{k+1})_{\geq 0}(x)$ for negative x on the equator. Hence $\phi \circ (\alpha_{k+1})_{\geq 0}, \phi \circ (\alpha_{k+1})_{\leq 0}$ agree on the equator, since $\phi \circ \rho^2 = \sigma^2 \circ \phi = \phi$. Moreover, both composites are continuous; for the second, we have

$$\phi \circ (\alpha_{k+1})_{\leq 0} = \phi \circ \rho \circ \widetilde{\alpha}_k \circ \pi \circ (-) = \sigma \circ (\phi \circ \widetilde{\alpha}_k) \circ \pi \circ (-)$$

and $\sigma, \phi \circ \widetilde{\alpha}_k, \pi, (-)$ are continuous. Hence (α-4) holds.

Before showing (α-2), we show that ($\widetilde{\alpha}$-2) implies

$$\widetilde{\alpha}_k(x) \preceq \rho(\widetilde{\alpha}_k(-x))$$

for all $x \in B^{k+1}$ not on the equator. For such x, $u(-x)$ is on the open upper hemisphere and hence is positive. By ($\widetilde{\alpha}$-2), we have

$$\widetilde{\alpha}_k(x) \preceq \alpha_k(l(x)) = \alpha_k(-u(-x)) = \rho(\alpha_k(u(-x))) \preceq \rho(\widetilde{\alpha}_k(-x)).$$

This proves the inequality above.

Now we show (α-2). For $x \in S^{k+1}$ in the open upper hemisphere, we have

$$\alpha_{k+1}(x) = \widetilde{\alpha}_k(\pi(x)) \preceq \rho(\widetilde{\alpha}_k(-\pi(x))) = \rho(\widetilde{\alpha}_k(\pi(-x))) = \alpha_{k+1}(m(x))$$

by the inequality above. Hence (α-2) holds. □

Taking $\beta_n = \phi \circ \alpha_n$, Theorem 6.3.5 follows from the claims above. To see that β_n is a $\mathbb{Z}/2$-map, note that for $x \in S^n$ positive, we have

$$\beta_n(-x) = \phi(\alpha_n(-x)) = \phi(\rho(\alpha_n(x))) = \sigma(\phi(\alpha_n(x))) = \sigma(\beta_n(x))$$

The other conclusions of the theorem are clear. □

For a full lifted mpa, the preorder conditions of Theorem 6.3.5 are trivially satisfied, so we get:

Corollary 6.3.6. *Let Y, Z be topological spaces, equip Y with a $\mathbb{Z}$-action generated by $\rho\colon Y \to Y$, and equip Z with a $\mathbb{Z}/2$-action generated by $\sigma\colon Z \to Z$. Let $\phi\colon Y \to Z$ be continuous and equivariant, i.e., $\sigma \circ \phi = \phi \circ \rho$. If there is a full lifted mpa for (Y, Z, ϕ), then there exists a $\mathbb{Z}/2$-map $\beta_n\colon S^n \to Z$ for all integers $n \geq 0$.*

6.4 Constructing a lifted mpa

The goal of this section is to prove our main result, Theorem 6.1.3, by constructing a lifted mpa satisfying the conditions of Theorem 6.3.5. As a warm-up, we use Theorem 6.3.5 to prove the Hobby-Rice theorem, Theorem 6.1.1:

Proof of Theorem 6.1.1. The idea is to lift the space of functions with range in $\{\pm 1\}$ to nondecreasing functions with range in $\mathbb{Z}$. By describing a continuous map from pairs of such functions to paths between them, we will produce a lifted mpa, which will imply the result by Theorem 6.3.5.

Let Y be the space of nondecreasing functions $g\colon [0,1] \to \mathbb{Z}$ with finite range, and let Z be the space of functions $h\colon [0,1] \to \{\pm 1\}$. Equip Y, Z with the L^1-norm, and define $\rho(g) = g + 1$, $\sigma(h) = -h$, and

$$\phi(g)(x) = \begin{cases} 1 & g(x) \text{ even} \\ -1 & g(x) \text{ odd} \end{cases}$$

Let $g_0 \preceq g_1$ if $g_0(x) \leq g_1(x)$ for all $x \in [0,1]$. Finally, for $g_0 \preceq g_1$ define $s_w(g_0, g_1)$ to be the path (in t) of functions following g_0 on $[0, 1-t)$ and g_1 on $[1-t, 1]$ (independent of w):

$$s_w(g_0, g_1)(t)(x) = \begin{cases} g_0(x) & x < 1 - t \\ g_1(x) & x \geq 1 - t \end{cases}$$

This definition gives $s_w(g_0, g_1)(0) = g_0$ and $s_w(g_0, g_1)(1) = g_1$. The conditions of Theorem 6.3.5 are straightforward to check, except perhaps the continuity property in the lifted mpa definition, which we check now.

We are given $g \in Y$, and we may assume V is a basis set, so that V consists of all $h \in Z$ with $\|h - \phi(g)\| < \varepsilon$ for some $\varepsilon > 0$. By our choice of U we may ensure that $g_0, g_1 \in Y$ have the same parity as g except on a sets S_0, S_1 with $\mu(S_i) < \varepsilon/4$. Then functions g' along the path $s_w(g_0, g_1)$ have the same parity as g except on $S_0 \cup S_1$, where $\mu(S_0 \cup S_1) < \varepsilon/2$, which implies $\|\phi(g') - \phi(g)\| < \varepsilon$.

Hence the conditions of Theorem 6.3.5 are satisfied, so we obtain a $\mathbb{Z}/2$-map $\beta_n \colon S^n \to Z$. Applying the Borsuk–Ulam theorem to $\psi \circ \beta_n \colon S^n \to \mathbb{R}^n$, where $\psi \colon h \mapsto (\int_0^1 f_j(x)h(x)dx)_j$, we obtain $x \in S^n$ with $\psi(\beta_n(x)) = 0$. Hence also $\psi(\beta_n(-x)) = 0$, so we may assume x is positive. Taking $y^* = 0$ in the last part of Theorem 6.3.5, we may ensure that β_n maps each positive point of S^n to a point in Z of the form $\phi(g)$ with $0 \le g \le n$, so that $\phi(g)$ has at most n sign changes. ☐

Now we prove our main result, Theorem 6.1.3:

Proof of Theorem 6.1.3. Consider the space $C^\infty([0,1];\mathbb{R})$ with the L^1-norm, and let Y be the subspace of nondecreasing functions in $C^\infty([0,1];\mathbb{R})$, equipped with the action $\rho \colon g \mapsto g+\pi$. Let Z be $C^\infty([0,1];S^1)$ with the L^1-norm, equipped with the action $\sigma \colon h \mapsto -h$.

Define $\phi \colon Y \to Z$ by $\phi(g)(x) = e^{ig(x)}$; then ϕ is continuous since $x \mapsto e^{ix}$ is 1-Lipschitz:

$$\begin{aligned}\|\phi(g_2) - \phi(g_1)\|_1 &= \int_0^1 |e^{ig_2(x)} - e^{ig_1(x)}|dx \\ &\le \int_0^1 |g_2(x) - g_1(x)|dx \\ &\le \|g_2 - g_1\|_1.\end{aligned}$$

Define $(\preceq)$ on Y as $(\le)$ pointwise. Then properties (1) and (2) of Theorem 6.3.5 and the commutativity property $\phi \circ \rho = \sigma \circ \phi$ evidently hold.

It remains to construct the lifted mpa s. Let $\tau \colon \mathbb{R} \to [0,1]$ be a smooth, nondecreasing function with $\tau(x) = 0$ for $x \le -1$, and $\tau(x) = 1$ for $x \ge 1$. (For example, take an integral of a mollifier.) Then define $s_w \colon Y^2_\preceq \to PY$ by

$$s_w(g_0, g_1)(t)(x) = \left(1 - \tau\left(\frac{x-(1-t)}{w}\right)\right) g_0(x) + \tau\left(\frac{x-(1-t)}{w}\right) g_1(x).$$

Since τ is smooth, and since $x \mapsto (x-(1-t))/w$ is smooth for $w \ne 0$, the function

$s_w(g_0, g_1)(t) \colon [0, 1] \to \mathbb{R}$ is smooth. Also, $s_w(g_0, g_1)(t)$ is nondecreasing:

$$\begin{aligned}
&\frac{d}{dx}[s_w(g_0, g_1)(t)(x)] \\
&\quad = -\frac{1}{w} \cdot \tau' \left(\frac{x - (1-t)}{w} \right) \cdot g_0(x) + \left(1 - \tau \left(\frac{x - (1-t)}{w} \right) \right) \cdot g_0'(x) \\
&\qquad + \frac{1}{w} \cdot \tau' \left(\frac{x - (1-t)}{w} \right) \cdot g_1(x) + \tau \left(\frac{x - (1-t)}{w} \right) \cdot g_1'(x) \\
&\quad \geq \frac{1}{w} \cdot \tau' \left(\frac{x - (1-t)}{w} \right) \cdot (g_1(x) - g_0(x)) \\
&\quad \geq 0.
\end{aligned}$$

Therefore, $s_w(g_0, g_1)$ takes values in PY. Since $g_0 \leq g_1$, we have

$$g_0 \leq s_w(g_0, g_1)(t) \leq g_1,$$

so property (3) of Theorem 6.3.5 holds.

Next we show $s_w(g_0, g_1)(t)$ is continuous in w, g_0, g_1, t. First we establish a helpful result. Let B be the subspace of $L^\infty([0, 1]; \mathbb{R})$ consisting of smooth functions, and let $\widetilde{Y}$ be the space $L^1([0, 1]; \mathbb{R})$, of which Y is a subspace; then pointwise multiplication $(b, g) \mapsto b \cdot g$ defines a continuous map $B \times \widetilde{Y} \to \widetilde{Y}$, via the following inequality, using Hölder's inequality:

$$\begin{aligned}
\|b_2 g_2 - b_1 g_1\|_1 &\leq \|b_2(g_2 - g_1)\|_1 + \|g_1(b_2 - b_1)\|_1 \\
&\leq \|b_2\|_\infty \cdot \|g_2 - g_1\|_1 + \|g_1\|_1 \cdot \|b_2 - b_1\|_\infty.
\end{aligned}$$

Since $(w, g_0, g_1, t) \mapsto g_0$, $(w, g_0, g_1, t) \mapsto g_1$ are continuous maps $(0, 1] \times Y \times Y \times [0, 1] \to Y$, by the result above it suffices to show that

$$(w, g_0, g_1, t) \mapsto \left(x \mapsto \tau \left(\frac{x - (1-t)}{w} \right) \right)$$

is a continuous map to B; the subtraction from 1 in the first term is handled by virtue of the fact that B is a normed linear space, so that pointwise addition and scalar multiplication by -1 each define a continuous map.

Since τ is constant outside of the compact set $[-1, 1]$, τ is uniformly continuous, hence it suffices to prove that

$$(w, g_0, g_1, t) \mapsto \left(x \mapsto \frac{x - (1-t)}{w} \right)$$

is a continuous map to B. Note that

$$\sup_{x \in [0,1]} \left| \frac{x}{w_2} - \frac{x}{w_1} \right| = \left| \frac{1}{w_2} - \frac{1}{w_1} \right|$$

Since $w \mapsto 1/w$ is a continuous map $\mathbb{R} \setminus \{0\} \to \mathbb{R}$, the map $(w, g_0, g_1, t) \mapsto (x \mapsto x/w)$ is a continuous map to B, as is $(w, g_0, g_1, t) \mapsto (x \mapsto -(1-t)/w)$, so the map above is indeed a continuous map to B. Hence $s_w(g_0, g_1)(t)$ is continuous in w, g_0, g_1, t.

It remains to show the continuity property for a lifted mpa. Let $g \in Y$, then for $g_0, g_1 \in Y$ we have

$$\begin{aligned}
&\|\phi(s_w(g_0, g_1)(t)) - \phi(g)\|_1 \\
&\qquad = \int_0^{1-t-w} |\phi(g_0)(x) - \phi(g)(x)|dx + \int_{1-t+w}^1 |\phi(g_1)(x) - \phi(g)(x)|dx \\
&\qquad\quad + \int_{1-t-w}^{1-t+w} |\phi(s_w(g_0, g_1)(t))(x) - \phi(g)(x)|dx \\
&\qquad \leq \|\phi(g_0) - \phi(g)\|_1 + \|\phi(g_1) - \phi(g)\|_1 + 4w,
\end{aligned}$$

where we use the fact that S^1 has diameter 2 in the last step. This inequality implies the continuity property for a lifted mpa.

Therefore, we may apply Theorem 6.3.5 to obtain a $\mathbb{Z}/2$-map $\beta_n \colon S^n \to Z$. Then $\psi \circ \beta_n \colon S^n \to \mathbb{R}^n$ is a $\mathbb{Z}/2$-map, so by the Borsuk–Ulam theorem, we have $\psi(\beta_n(x)) = 0$ for some $x \in S^n$, and we may assume x is positive. Taking $y^* = c_0$ in the last part of Theorem 6.3.5, we have $\rho^n(y^*) = c_n$, so we may ensure that $h = \beta_n(x)$ is of the form $\phi(g)$ for $g \in Y$, where g is an increasing function with range in $[0, \pi n]$. This gives the desired $W^{1,1}$-norm bound:

$$\int_0^1 \left| \frac{d}{dx}[e^{ig(x)}] \right| dx = \int_0^1 |g'(x)|dx = g(1) - g(0) \leq \pi n,$$

which implies $\|h\|_{W^{1,1}} \leq 1 + \pi n$. □

6.5 Improving the bound further

In the introduction we argued that a $W^{1,1}$-norm bound of $1 + 2\pi n$ in Theorem 6.1.2 might be expected from smoothing the Hobby–Rice theorem. In this section, we show an improved bound for Theorem 6.1.2 in the case where the f_j are real-valued. The idea is to modify the S^1 step of our construction so that some functions in the image of α_k have smaller range within $[0, \pi k]$, and to modify the later steps so that functions h in the image of α_k with large range have $\psi(\phi(h)) \neq 0$.

Theorem 6.5.1. *Let $f_1, \ldots, f_n \in L^1([0,1]; \mathbb{R})$. Then there exists $h \in C^\infty([0,1]; S^1)$ such that for all j,*

$$\int_0^1 f_j(x)h(x)dx = 0.$$

Moreover, for any $\varepsilon > 0$, h can be chosen such that

$$\|h\|_{W^{1,1}} < 1 + \pi(2n-1) + \varepsilon.$$

Proof. Define $Y, Z, \rho, \sigma, \phi, s$ as in the proof of Theorem 6.1.3, let $y^* = c_0$, and let $(\preceq)$ be $(\leq)$. We will produce $\alpha_n\colon S^n \to Y$ and $\beta_n\colon S^n \to Z$ by the inductive construction in the proof of Theorem 6.3.5, but we modify the first step by defining $\alpha_1\colon S^1 \to Y$ by $e^{ix} \mapsto c_x$ for $x \in [0, 2\pi)$. This α_1 differs from the α_1 obtained in the proof of Theorem 6.3.5, which only gives constant functions at $\pm 1 \in S^1$, but is still good in the sense introduced in the proof of Theorem 6.3.5. Using this α_1 as our base case, we inductively construct α_k as before with the following additional condition:

For $\delta > 0$ (depending on k and the f_j), α_k may be chosen such that for all x:

$$\text{Re}[e^{i\alpha_k(x)(t)}] = \pi_1(x) \qquad \text{for } t \in [0,1] \setminus S, \text{ where } \mu_f(S) < \delta \qquad (P_{\alpha_k,\delta})$$

Here μ_f is as in the proof of Corollary 6.1.4, that is,

$$\mu_f(S) = \int_0^1 |f_j(x)| dx,$$

and $\pi_1\colon S^k \to [-1, 1]$ is the projection to the first coordinate.

The condition $(P_{\alpha_k,\delta})$ holds for $k = 1$ and all $\delta > 0$ by our definition of α_1. To show that the condition carries through the inductive step, it suffices to show that given $\delta > 0$, there exists $\delta' > 0$ such that given α_k such that $(P_{\alpha_k,\delta'})$ holds, we can extend α_k to $\widetilde{\alpha}_k$ as in the first claim in the proof of Theorem 6.3.5 such that $(P_{\widetilde{\alpha}_k,\delta})$ holds.

We accomplish this by modifying the definition of $\widetilde{\alpha}_k$ in the first claim in the proof of Theorem 6.3.5 to impose a universal upper bound on $w(x)$. Since μ_f is absolutely continuous with respect to Lebesgue measure λ, for $\delta'' > 0$ there exists $\delta''' > 0$ such that $\lambda(S) \leq 2\delta'''$ implies $\mu_f(S) < \delta''$. Then we use δ''' as our upper bound on $w(x)$:

$$\widetilde{\alpha}_k(x) = \begin{cases} \alpha_k(x) & x \in E \\ s_{w(x)}(\alpha_k(u(x)), \alpha_k(l(x)))(t(x)) & x \notin E \end{cases}$$

$$\text{where } w(x) = \min(d(x, E), t(x), 1 - t(x), \delta''')$$

$$t(x) = \frac{d(u(x), x)}{d(u(x), l(x))}$$

This ensures that functions in the image of $\widetilde{\alpha}_k$ are equal to one of the functions $\alpha_k(u(x)), \alpha_k(l(x))$ except on a set S with $\mu_f(S) < \delta''$. Hence we may take $\delta' = \delta'' = \delta/2$; then $(P_{\widetilde{\alpha_k}}, \delta)$ holds as desired. This shows that for any $\delta > 0$, α_k may be chosen such that $(P_{\alpha_k,\delta})$ holds.

Now we apply the Borsuk–Ulam theorem as before. We have the following diagram:

$$S^{2n} \xrightarrow[(\mathbb{Z}/2)]{\phi \circ \alpha_{2n}} Z \xrightarrow[(\mathbb{Z}/2)]{\psi} \mathbb{C}^n$$

The composition $\psi \circ \phi \circ \alpha_{2n}$ is a $\mathbb{Z}/2$-map, so the Borsuk–Ulam theorem implies that it has a zero; that is, there exists $x \in S^{2n}$ such that for all j, we have

$$\int_0^1 f_j(t) e^{i\alpha_{2n}(x)(t)} dt = 0.$$

Moreover, we may assume $x \in S^{2n}$ is positive.

But by the above, we have for the real parts, for all j,

$$\begin{aligned}
&\Re\left[\int_0^1 f_j(t)e^{i\alpha_{2n}(x)(t)}dt\right] \\
&\quad = \int_0^1 f_j(t)\cdot \Re[e^{i\alpha_{2n}(x)(t)}]dt \\
&\quad = \pi_1(x)\cdot \int_0^1 f_j(t)dt + \int_S f_j(t)(\Re[e^{i\alpha_{2n}(x)(t)}] - \pi_1(x))dx.
\end{aligned}$$

We can bound the last term as follows:

$$\begin{aligned}
\left|\int_S f_j(t)(\Re[e^{i\alpha_{2n}(x)(t)}] - \pi_1(x))dx\right| &\le \int_S |\Re[e^{i\alpha_{2n}(x)(t)}] - \pi_1(x)|d\mu_f \\
&\le 2\mu_f(S).
\end{aligned}$$

Now if all $\int_0^1 f_j(t)dt$ are 0, then we may take h to be an arbitrary constant, which gives $\|h\|_{W^{1,1}} = 1$. Hence we may assume that some $\int_0^1 f_j(t)dt$ is nonzero. In this case, we may ensure that for the x with $(\psi\circ\phi\circ\alpha_{2n})(x) = 0$ guaranteed by the Borsuk–Ulam theorem, $\pi_1(x)$ is smaller than any constant we like, by taking δ small in $(P_{\alpha_{2n},\delta})$. In particular, choose δ sufficiently small such that $|\Re[e^{i\theta}]| < \delta$ implies $|\theta - \pi/2| < \varepsilon'$ for $\theta \in [0,\pi]$.

Now we analyze the ranges of functions $\alpha_k(x)\colon [0,1] \to \mathbb{R}$ with x positive and $|\pi_1(x)| < \delta$, using the fact that functions $\alpha_{k+1}(x)$ are produced as transition functions between two functions $\alpha_k(x'), \alpha_k(x'')$ with $\pi_1(x') = \pi_1(x'') = \pi_1(x)$. For $k = 1$, $\alpha_k(x)$ has range in $[\pi/2 - \varepsilon', \pi/2 + \varepsilon']$, and each increment of k extends the right end of this interval by π. Hence $\alpha_{2n}(x)$ has range in

$$[\pi/2 - \varepsilon', \pi/2 + \pi(2n-1) + \varepsilon'].$$

Hence taking $h = \phi(\alpha_{2n}(x))$ gives $\|h\|_{W^{1,1}} \le 1 + \pi(2n-1) + 2\varepsilon'$. Choosing $\varepsilon' < \varepsilon/2$ gives the desired result. □

6.6 A lower bound

We ask whether $\|h\|_{W^{1,1}} \le 1 + 2n\pi$ is the best possible bound in Theorem 6.1.2. We prove a lower bound of $1 + n\pi$ in the case that the f_j are real-valued, which implies the same lower bound in the case that the f_j are complex-valued.

Theorem 6.6.1. *There exist $f_1,\dots,f_n \in L^1([0,1];\mathbb{R})$, such that for any $h \in C^1([0,1];S^1)$ with*

$$\int_0^1 f_j(x)h(x)dx = 0 \qquad j = 1,\dots,n$$

we have $\|h\|_{W^{1,1}} > \pi n + 1$.

Proof. Consider the case $n = 1$, and take f_1 constant and nonzero. Suppose for contradiction that $\|h\|_{W^{1,1}} \leq \pi + 1$, and write $h(x)$ as $e^{ig(x)}$ for $g \in C^1([0,1];\mathbb{R})$, so that $\int_0^1 |g'(x)|dx \leq \pi$. Since g is continuous, g attains its minimum m and maximum M on $[0,1]$. By adding a constant to g, we may assume $m = 0$; then we have $M \leq \pi$.

Since f_1 is constant, we have $\int_0^1 h(x)dx = 0$, so $\int_0^1 \mathrm{Im}(h(x))dx = 0$. But $\mathrm{Im}(h(x))$ is continuous in x and nonnegative, so $\mathrm{Im}(h(x)) = 0$ for all x. Hence h is constant at either 1 or -1, but this contradicts $\int_0^1 h(x)dx = 0$. Therefore, $\|h\|_{W^{1,1}} > \pi + 1$ for $n = 1$.

Now allow n arbitrary, and take each f_j to be the indicator function on a disjoint interval I_j. If $\|h\|_{W^{1,1}} \leq \pi n + 1$, then $\int_{I_j} |g'(x)|dx \leq \pi$ for some j, and we obtain a contradiction as above. Therefore, $\|h\|_{W^{1,1}} > \pi n + 1$. □

This $W^{1,1}$-norm bound establishes an upper bound for the coindex of the space of smooth circle-valued functions with norm at most $1 + \pi n$:

Theorem 6.6.2. *For integer $n \geq 1$ let Y_n denote the space of C^∞-functions $f\colon [0,1] \to S^1$ with $\|f\|_{W^{1,1}} \leq 1 + \pi n$. Then*

$$n \leq \mathrm{coind}\, Y_n \leq 2n - 1.$$

Proof. In the proof of Theorem 6.1.3 we constructed a $\mathbb{Z}/2$-map $\beta_n\colon S^n \to Y_n$, which shows that $\mathrm{coind}\, Y_n \geq n$. Let $f_1, \dots, f_n$ be chosen as in Theorem 6.6.1. Then the map $\psi\colon Y_n \to \mathbb{R}^{2n}$ given by $\psi(h) = (\int_0^1 f_j(x)h(x)dx)_j$ has no zero and is a $\mathbb{Z}/2$-map. Thus ψ radially projects to a $\mathbb{Z}/2$-map $Y_n \to S^{2n-1}$. A $\mathbb{Z}/2$-map $S^{2n} \to Y_n$ would compose with ψ to a $\mathbb{Z}/2$-map $S^{2n} \to S^{2n-1}$, contradicting the Borsuk–Ulam theorem. This implies $\mathrm{coind}\, Y_n \leq 2n - 1$. □

Problem 6.6.3. Determine the homotopy type of Y_n.

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References

[1] Noga Alon. “Splitting necklaces”. In: *Adv. Math.* 63.3 (1987), pp. 247–253.

[2] Jai Aslam et al. “Splitting loops and necklaces: variants of the square peg problem”. In: *Forum Math. Sigma* 8 (2020), Paper No. e5, 16.

[3] Michael Farber. “Topological complexity of motion planning”. In: *Discrete Comput. Geom.* 29.2 (2003), pp. 211–221.

[4] Charles R. Hobby and John R. Rice. “A moment problem in L_1 approximation”. In: *Proc. Amer. Math. Soc.* 16.4 (1965), pp. 665–670.

[5] Oleg Lazarev and Elliott H. Lieb. “A smooth, complex generalization of the Hobby–Rice theorem”. In: *Indiana Univ. Math. J.* 62.4 (2013), pp. 1133–1141.

[6] Elliott H. Lieb and Robert Schrader. “Current densities in density-functional theory”. In: *Phys. Rev. A* 88.3 (2013), p. 032516.

[7] James Munkres. *Topology*. Pearson Education, 2014.

[8] Allan Pinkus. “A simple proof of the Hobby–Rice theorem”. In: *Proc. Amer. Math. Soc.* 60.1 (1976), pp. 82–84.

[9] Vermont Rutherfoord. “On the Lazarev–Lieb extension of the Hobby–Rice theorem”. In: *Adv. Math.* 244 (2013), pp. 16–22.