COMP 355
Advanced Algorithms

Dijkstra’s Algorithm for Shortest Paths
Sections 4.4 (KT)

Rhodes College
—1848—

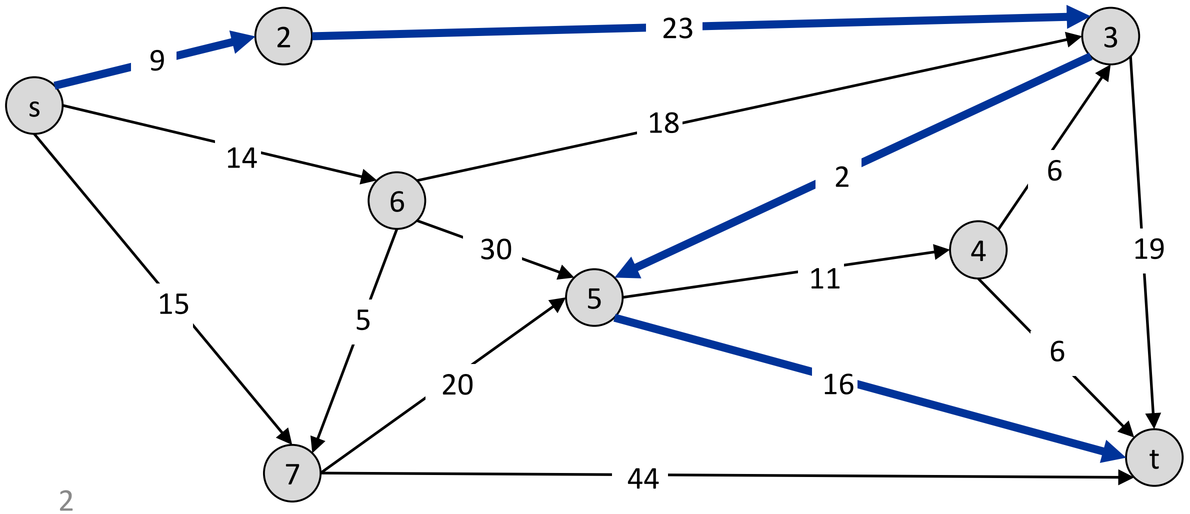
Shortest Path Problem

Shortest path network.

- Directed graph G = (V, E).
- Source s, destination t.
- Length ℓ_e = length of edge e.

Shortest path problem: find shortest directed path from s to t.

cost of path = sum of edge costs in path



Cost of path s-2-3-5-t
= 9 + 23 + 2 + 16
= 48.

Single Source Shortest Paths

Single Source Shortest Path Problem:

- Given a digraph $G = (V, E)$
- Numeric edge weights
- Source vertex, $s \in V$
- Determine the distance $\delta(s, v)$ from s to every vertex v in the graph

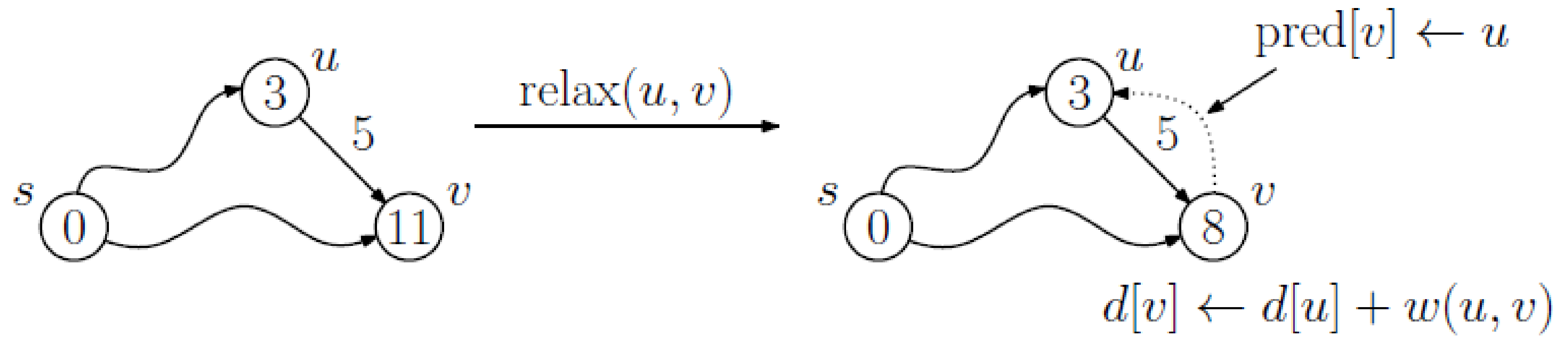
Are negative weight edges allowed? (could arise in financial transaction networks)

Dijkstra’s algorithm assumes no negative edge weights.

- Computing the distance from source to each vertex (not the actual path)

Shortest Paths and Relaxation

```
relax(u,v):
   if d[u] + w(u,v) < d[v]:        # is the path through u shorter?
      d[v] = d[u] + w(u,v)         # yes, then take it
      pred[v] = u                  # record that we go through u
```



Dijkstra's Algorithm

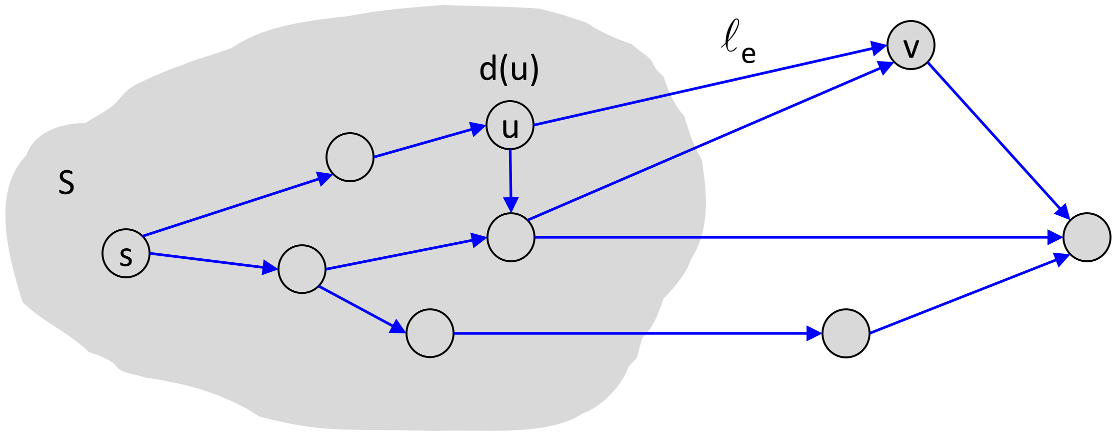
Dijkstra's algorithm.

- Maintain a set of explored nodes S for which we have determined the shortest path distance d(u) from s to u.
- Initialize S = { s }, d(s) = 0.
- Repeatedly choose unexplored node v which minimizes

$$\pi(v) = \min_{e = (u,v) : u \in S} d(u) + \ell_e ,$$

shortest path to some u in explored part, followed by a single edge (u, v)

add v to S, and set d(v) = π(v).



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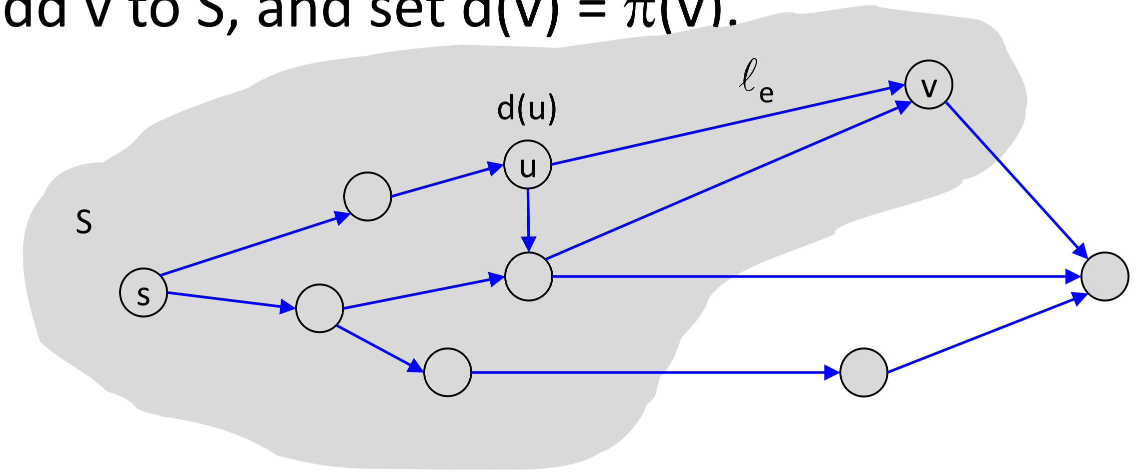


Dijkstra’s Algorithm: Implementation

Build: Create a priority queue from a list of n elements, each with an associated key value.

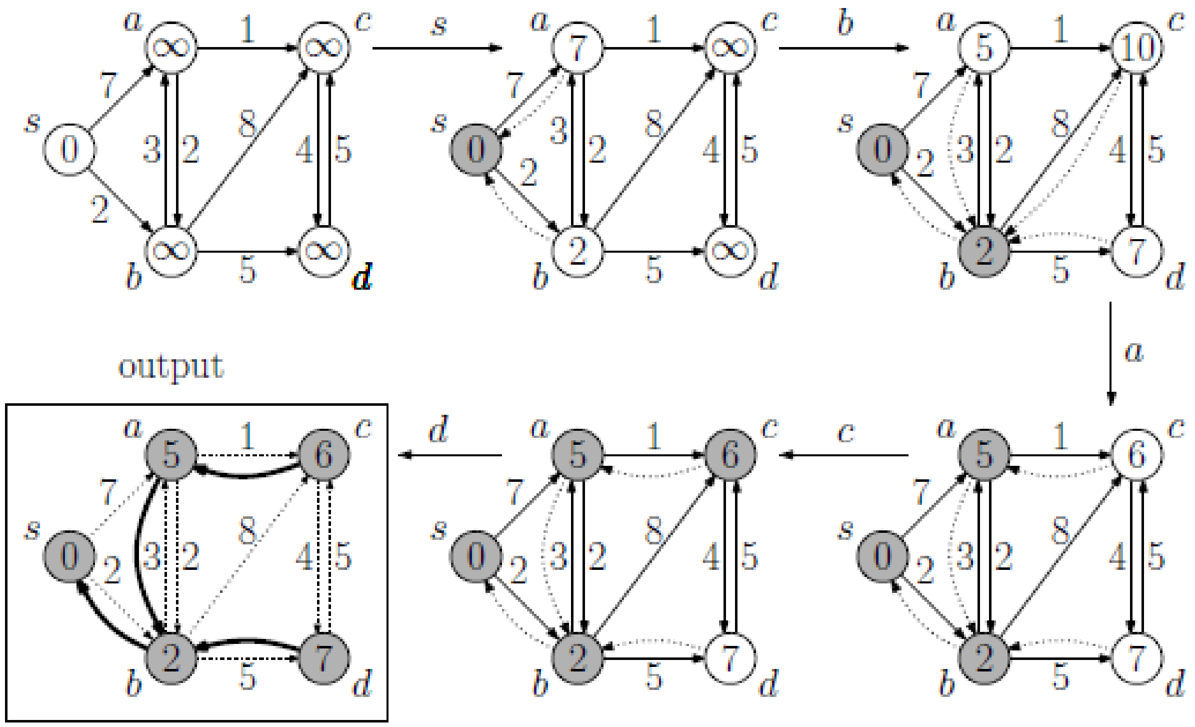
Extract min: Remove (and return a reference to) the element with the smallest key value.

Decrease key: Given a reference to an element in the priority queue, decrease its key value to a specified value, and reorganize if needed.

Dijkstra’s Algorithm

```
dijkstra(G,w,s) {
    for each (u in V) {                // initialization
        d[u] = +infinity
        mark[u] = undiscovered
        pred[u] = null
    }
    d[s] = 0                           // distance to source is 0
    Q = a priority queue of all vertices u sorted by d[u]
    while (Q is nonEmpty) {            // until all vertices processed
        u = extract vertex with minimum d[u] from Q
        for each (v in Adj[u]) {
            if (d[u] + w(u,v) < d[v]) { // relax(u,v)
                d[v] = d[u] + w(u,v)
                decrease v’s key in Q to d[v]
                pred[v] = u
            }
        }
        mark[u] = finished
    }
    [The pred pointers define an ‘‘inverted’’ shortest path tree]
}
```

Dijkstra’s Algorithm: Example

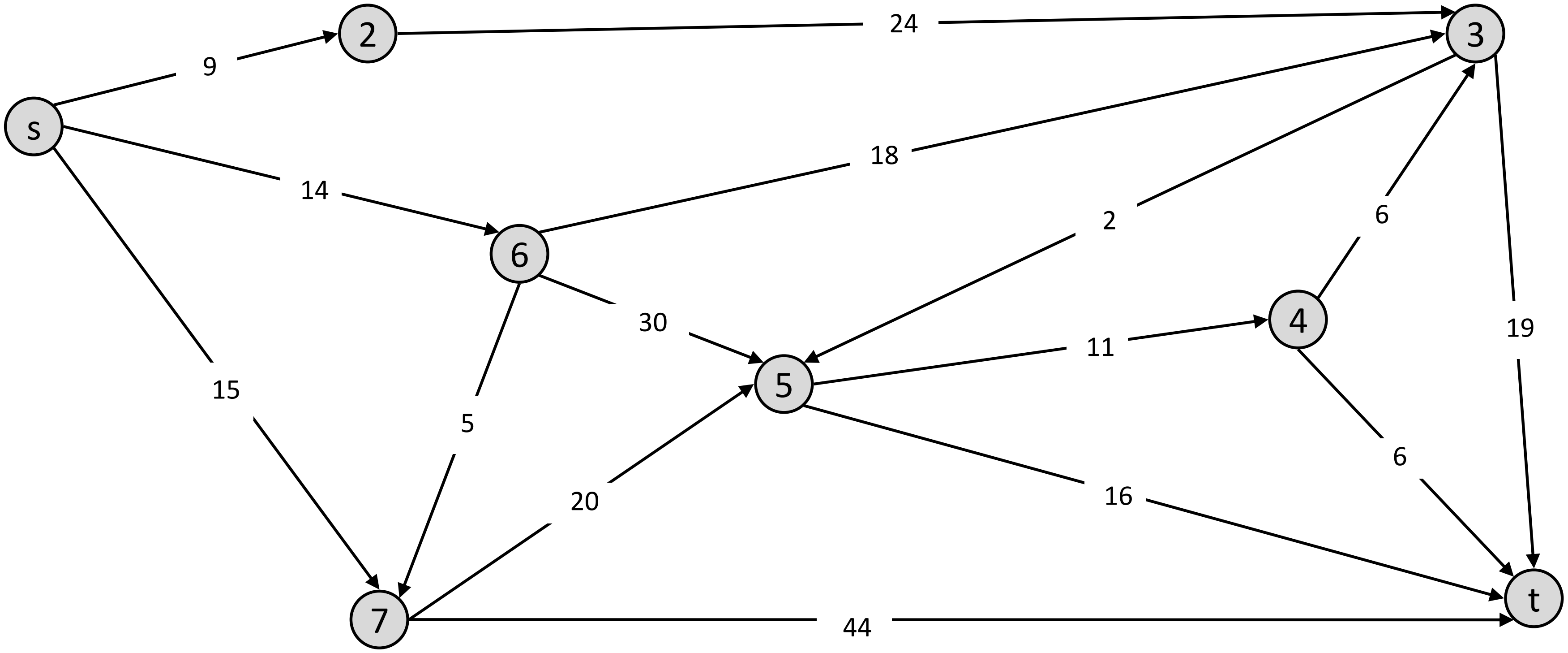


$$
\begin{aligned}
T(n,m) &= \sum_{u\in V}(\log n + \deg(u)\cdot\log n) = \sum_{u\in V}(1+\deg(u))\log n \\
&= \log n \sum_{u\in V}(1+\deg(u)) = (\log n)(n+2m) = \Theta((n+m)\log n).
\end{aligned}
$$

Since G is connected, n is asymptotically no greater than m, so this is $O(m \log n)$.

Dijkstra's Shortest Path Algorithm

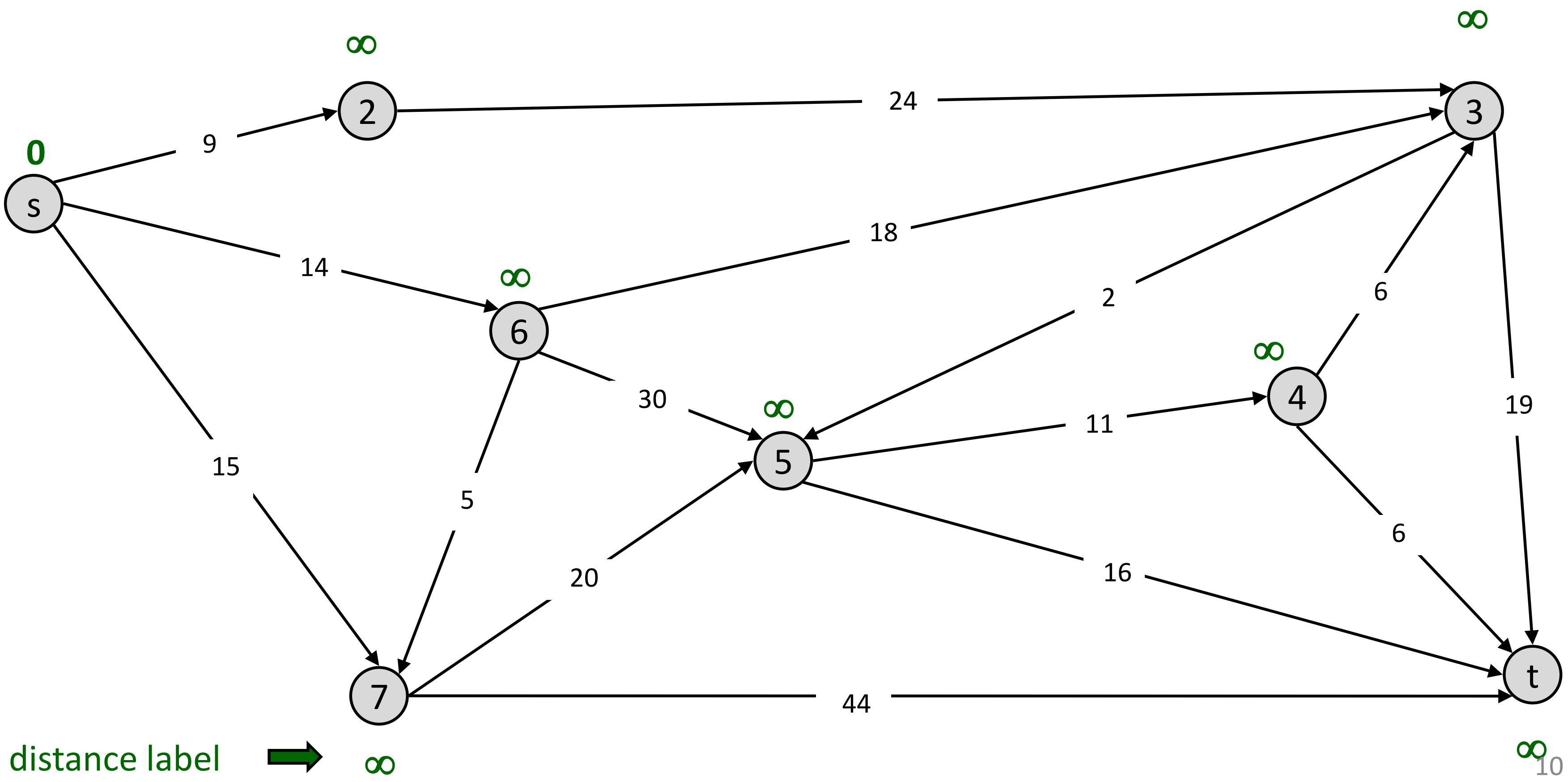
Find shortest path from s to t.



Dijkstra's Shortest Path Algorithm

S = { }

PQ = { s, 2, 3, 4, 5, 6, 7, t }



distance label ➡

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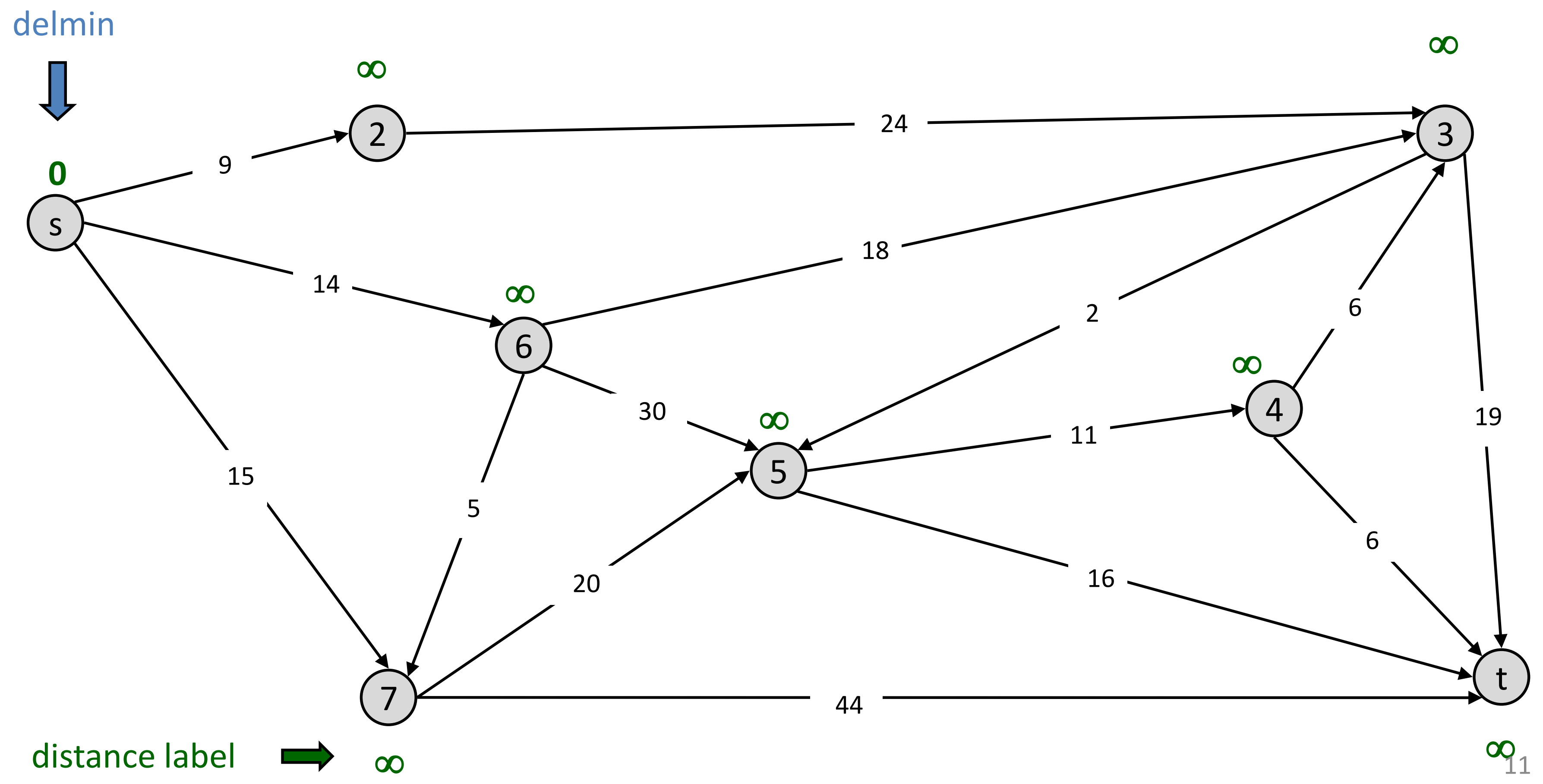

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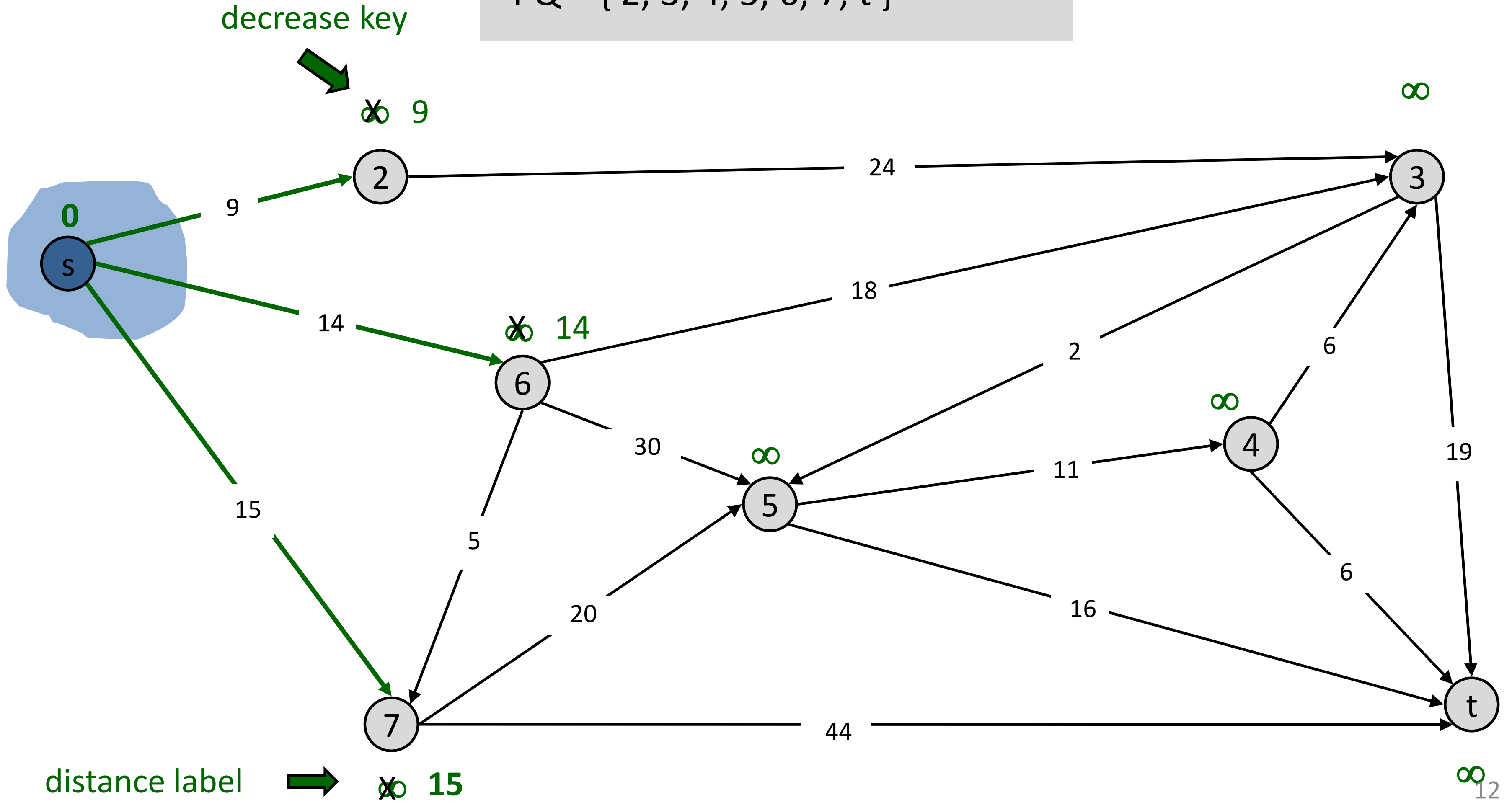

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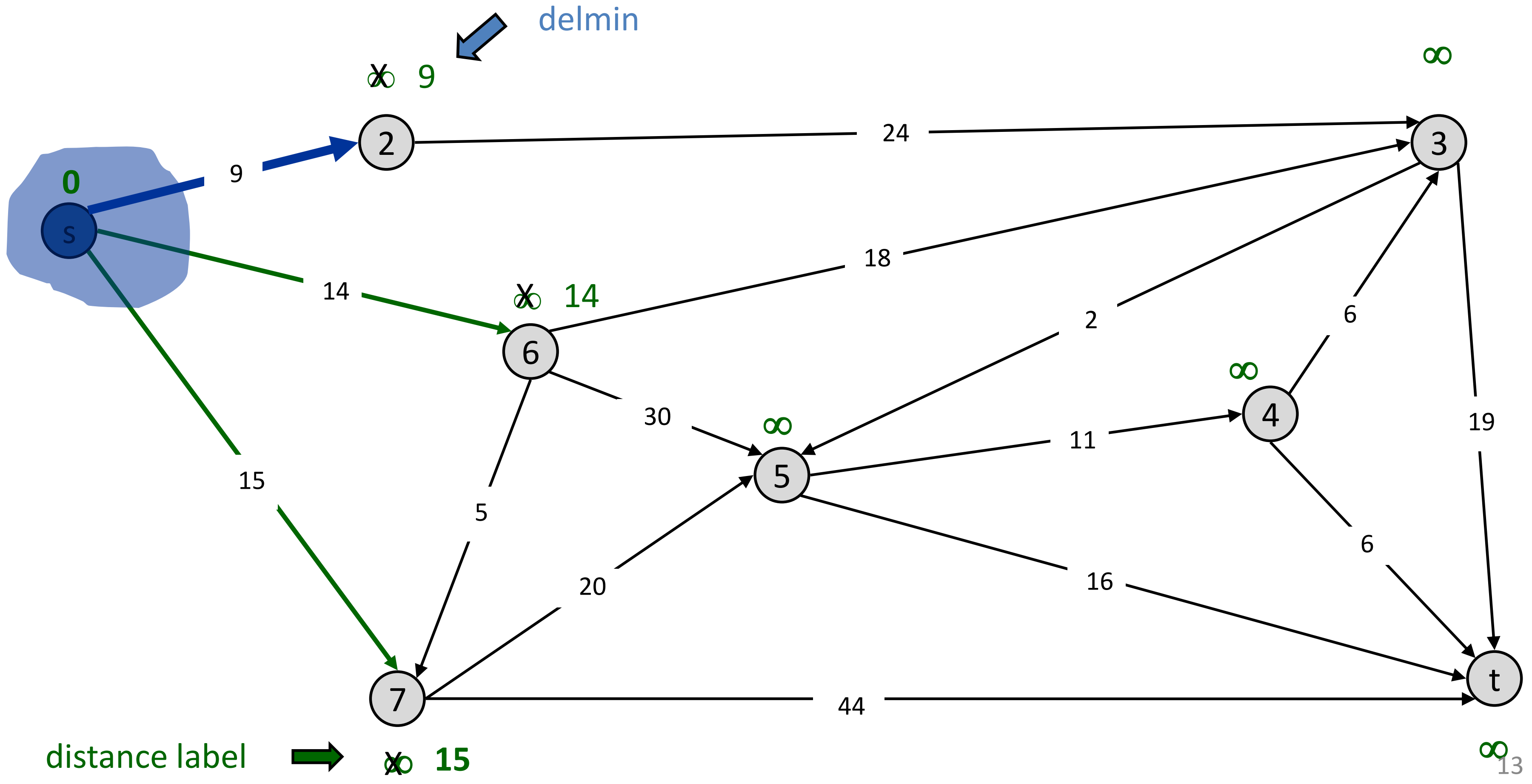
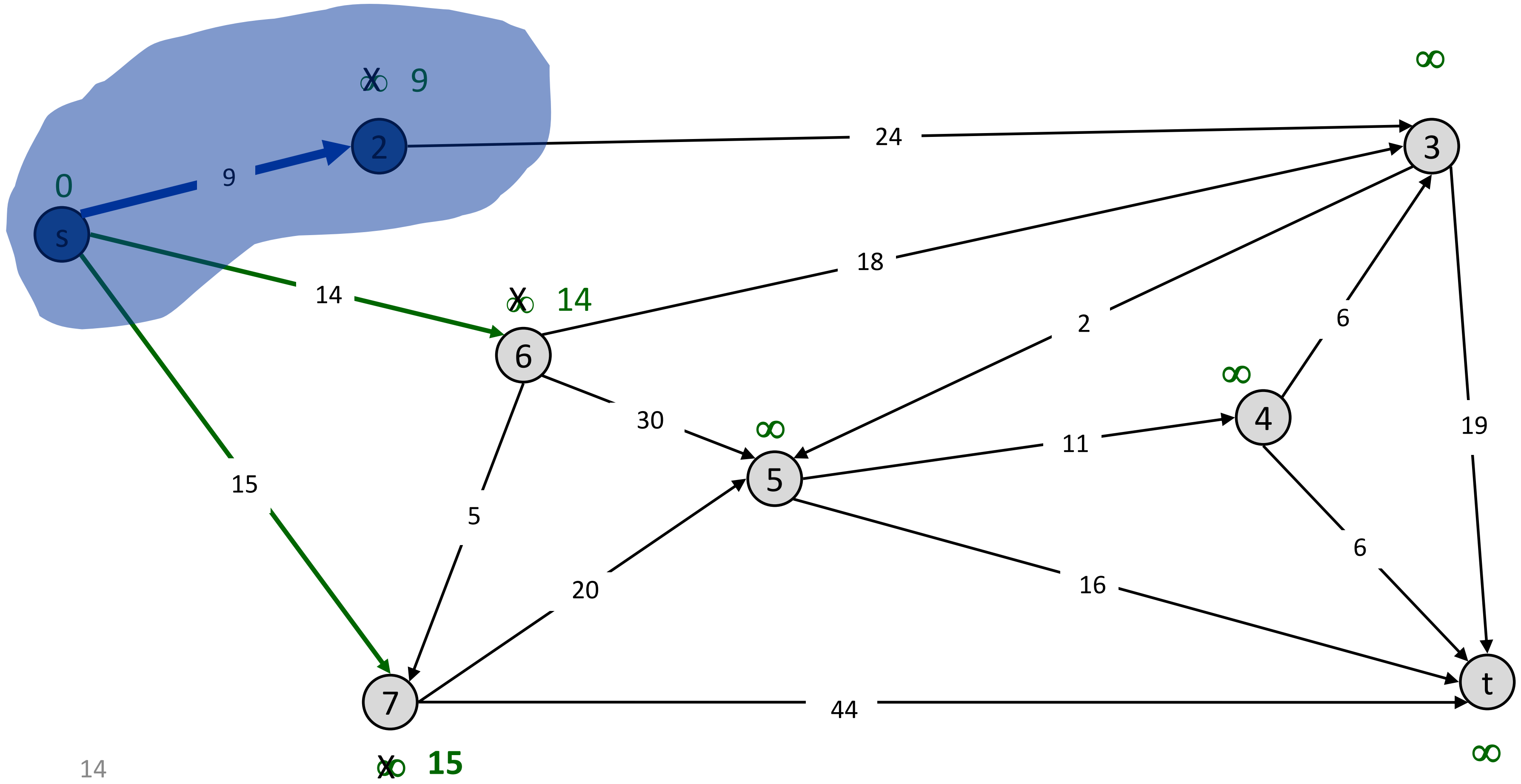


Dijkstra's Shortest Path Algorithm

S = { s, 2 }

PQ = { 3, 4, 5, 6, 7, t }



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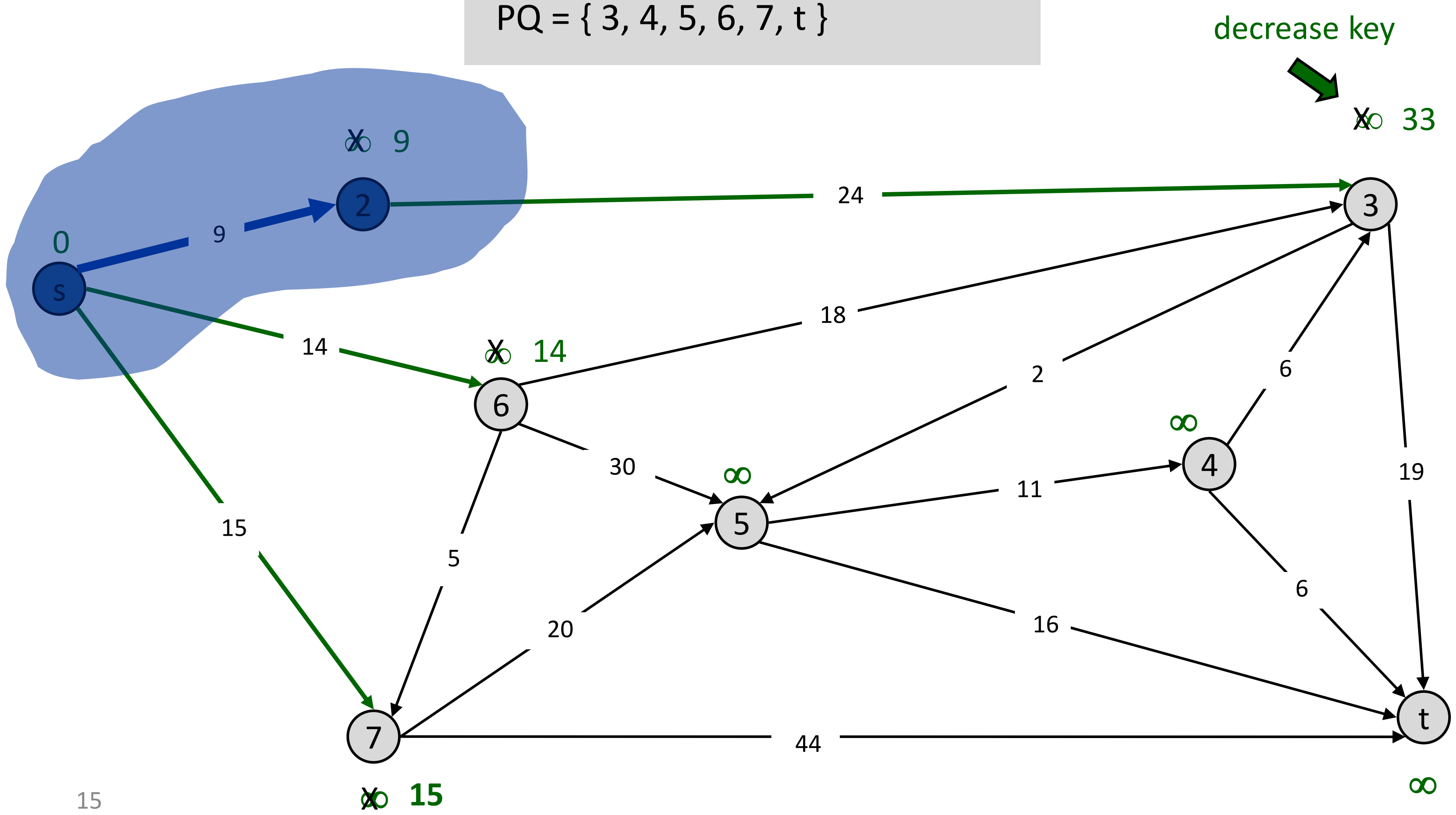

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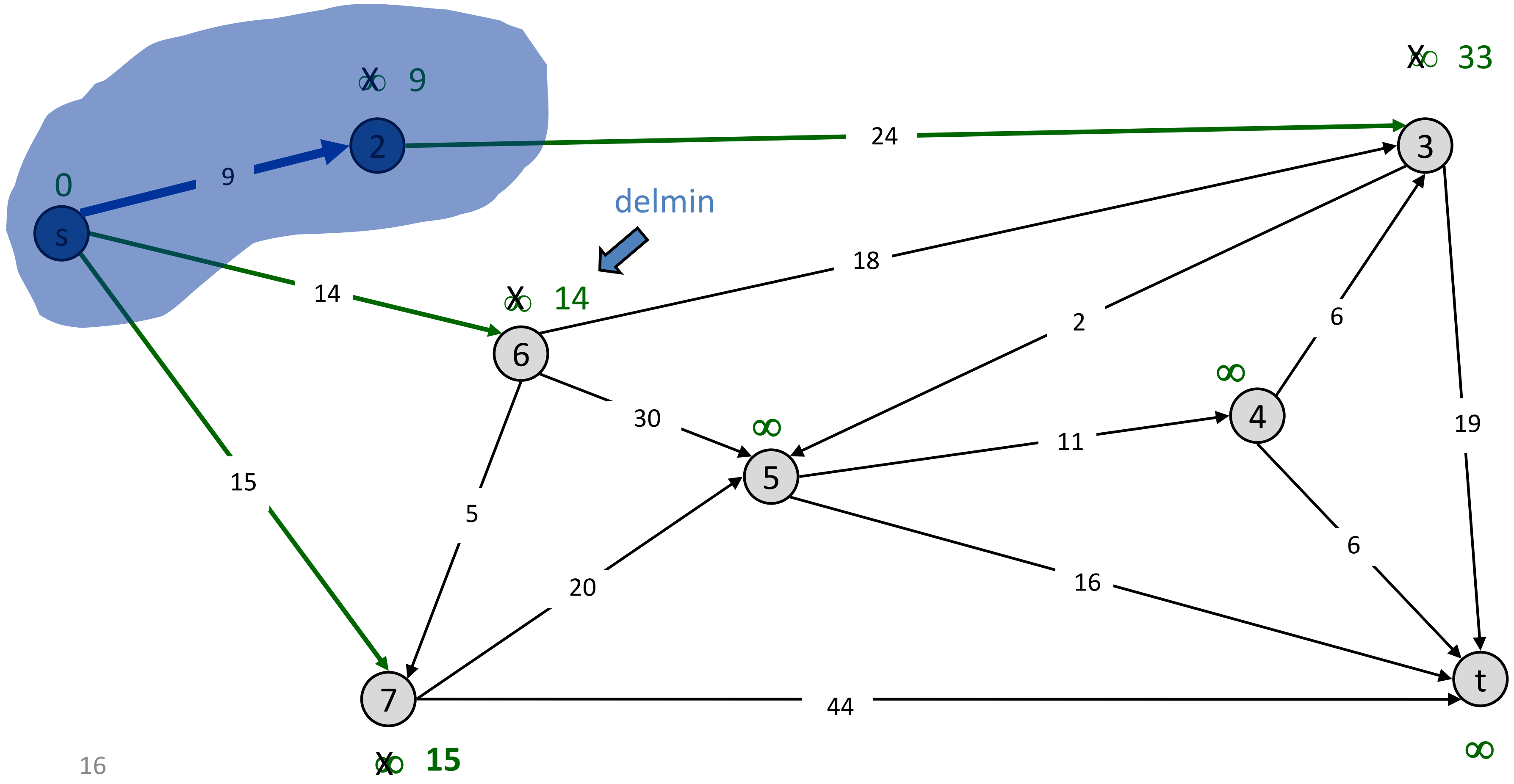
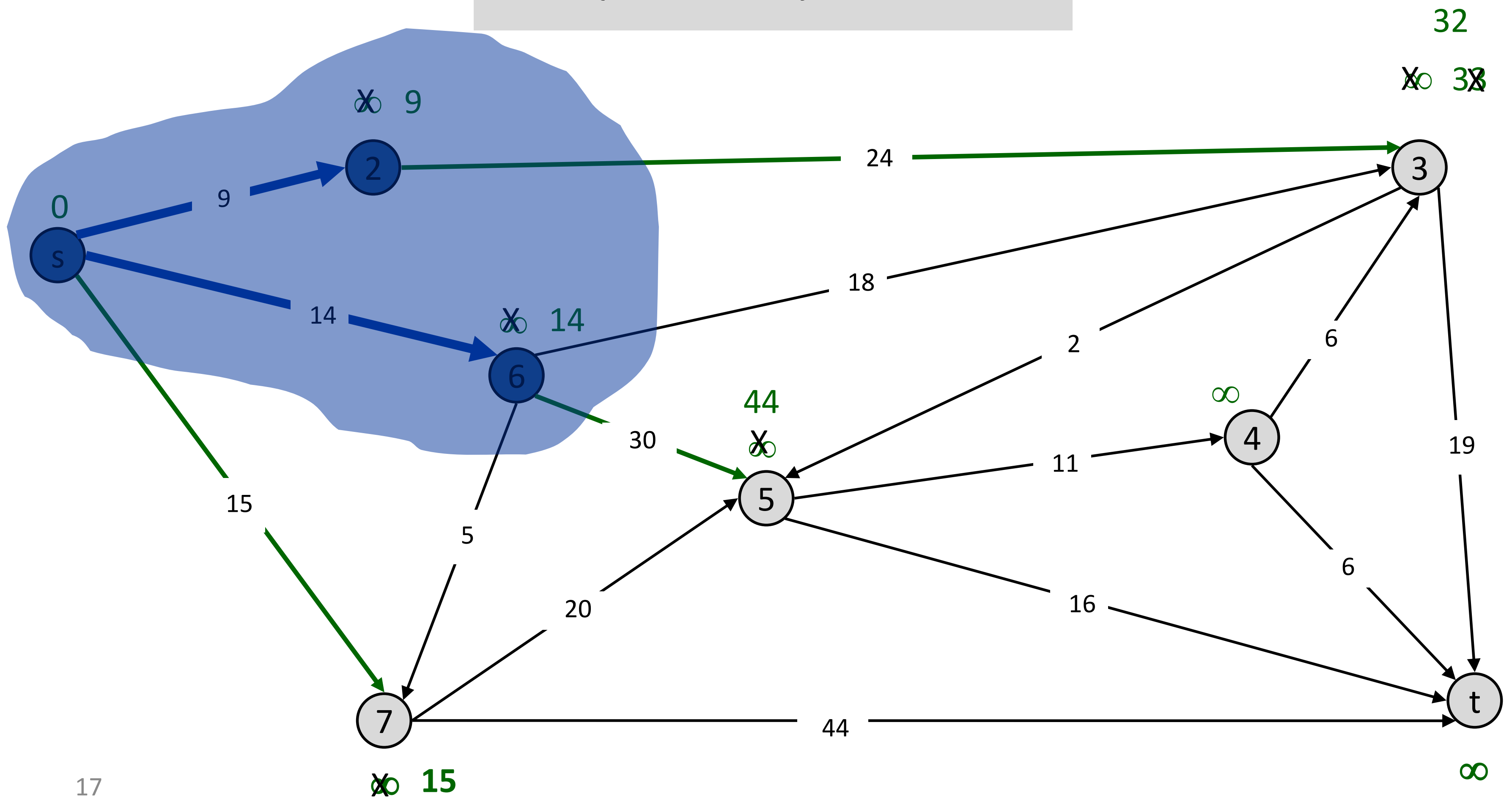


Dijkstra's Shortest Path Algorithm

S = { s, 2, 6 }

PQ = { 3, 4, 5, 7, t }



Dijkstra's Shortest Path Algorithm

S = { s, 2, 6 }

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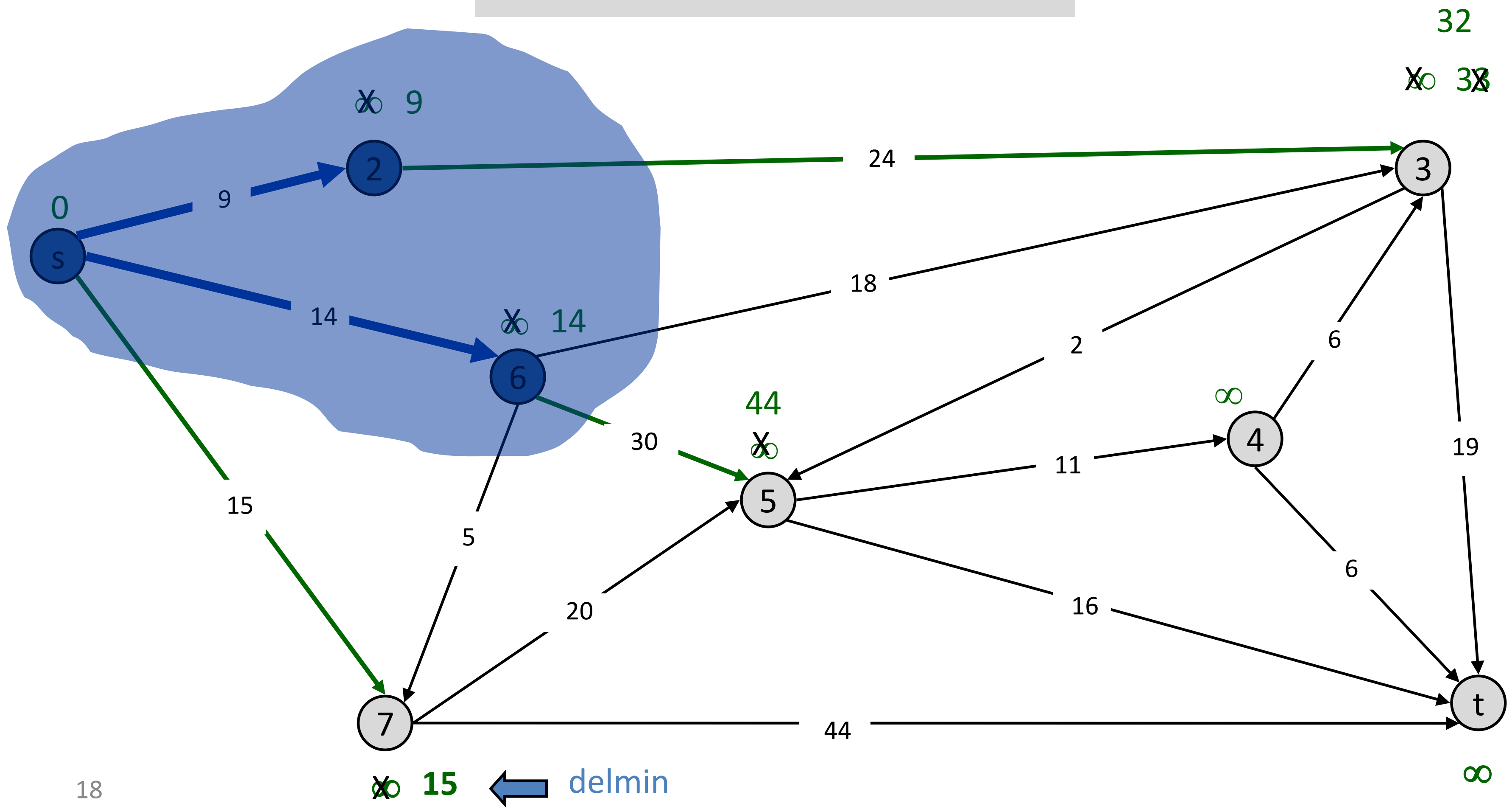

Dijkstra's Shortest Path Algorithm

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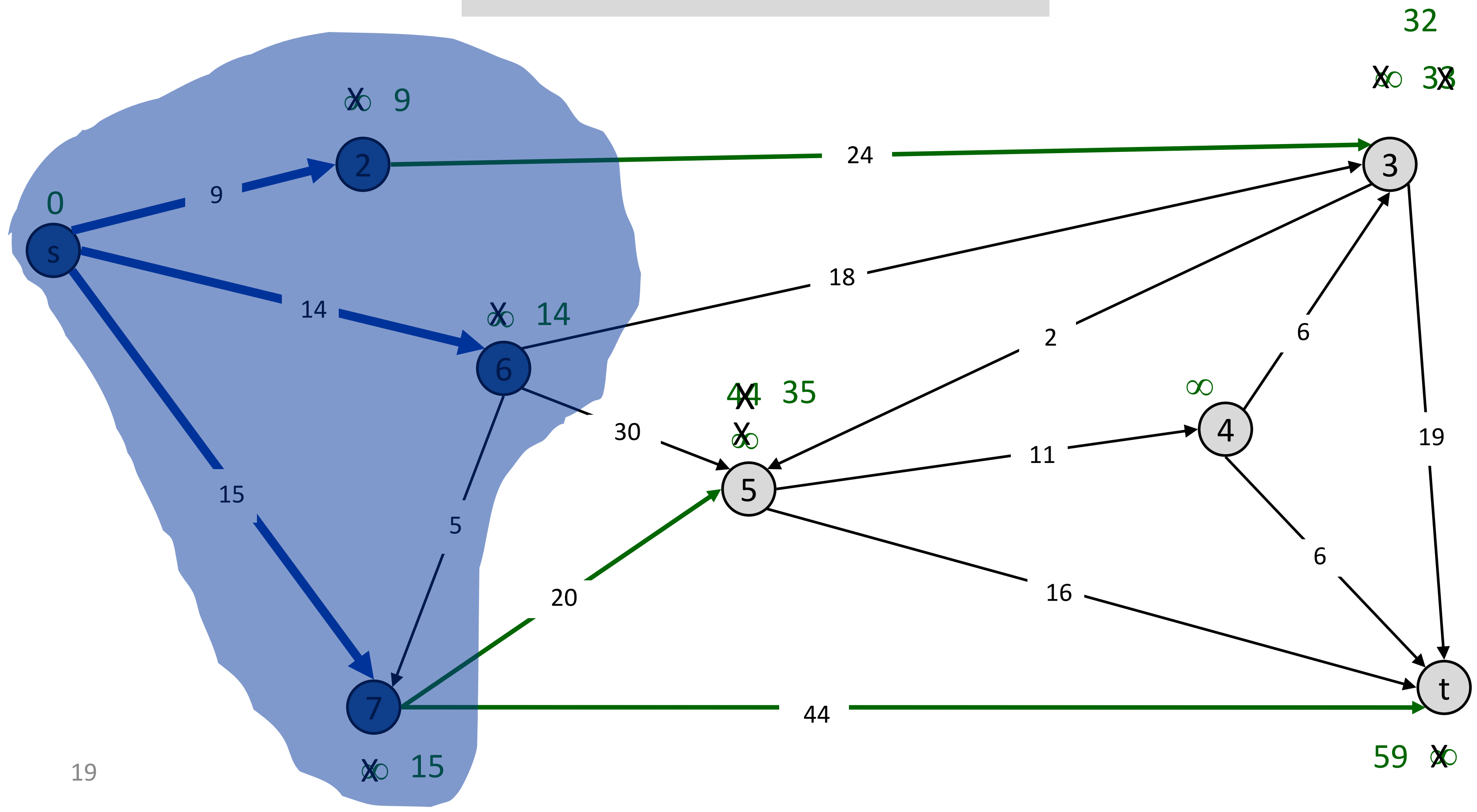

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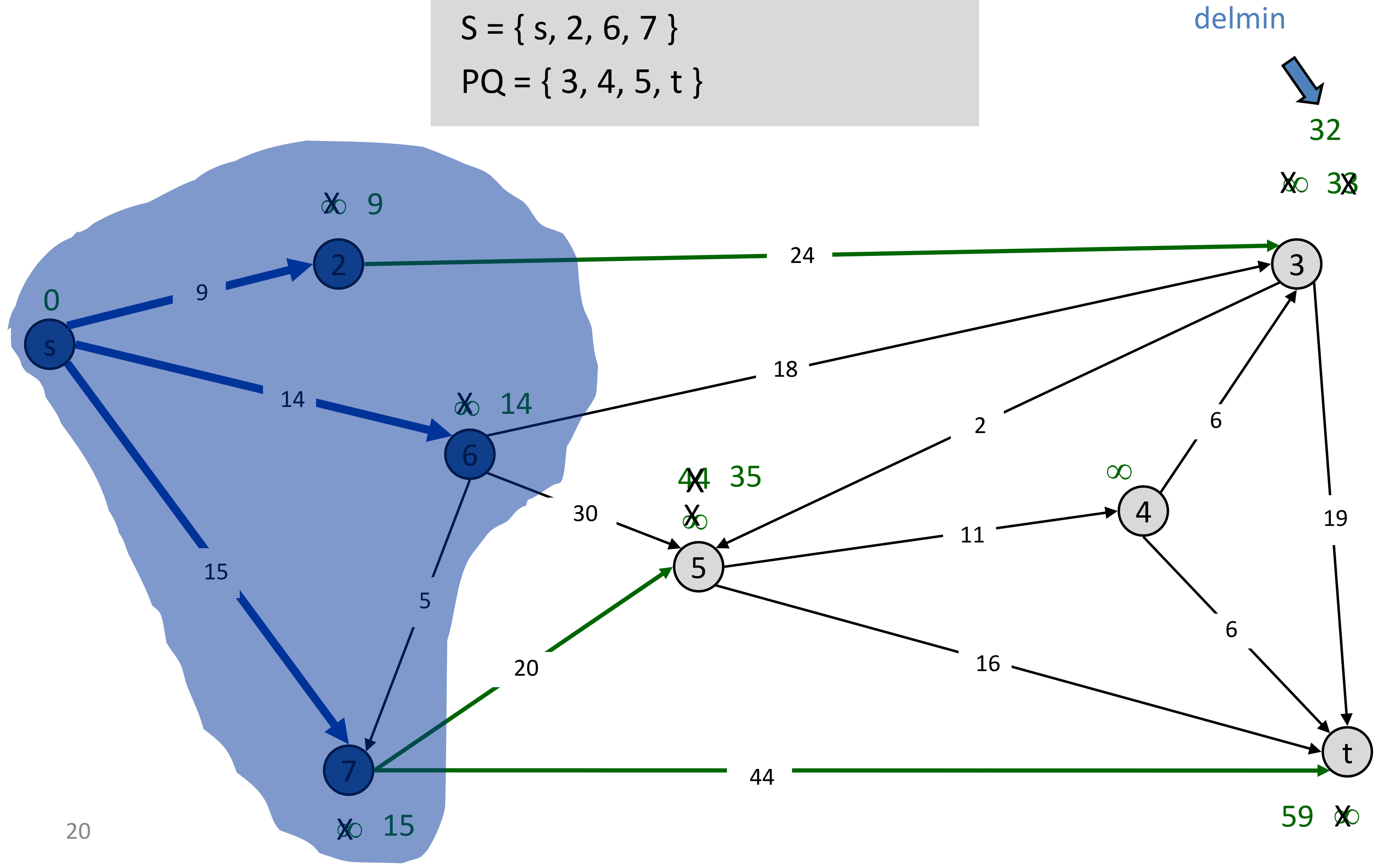

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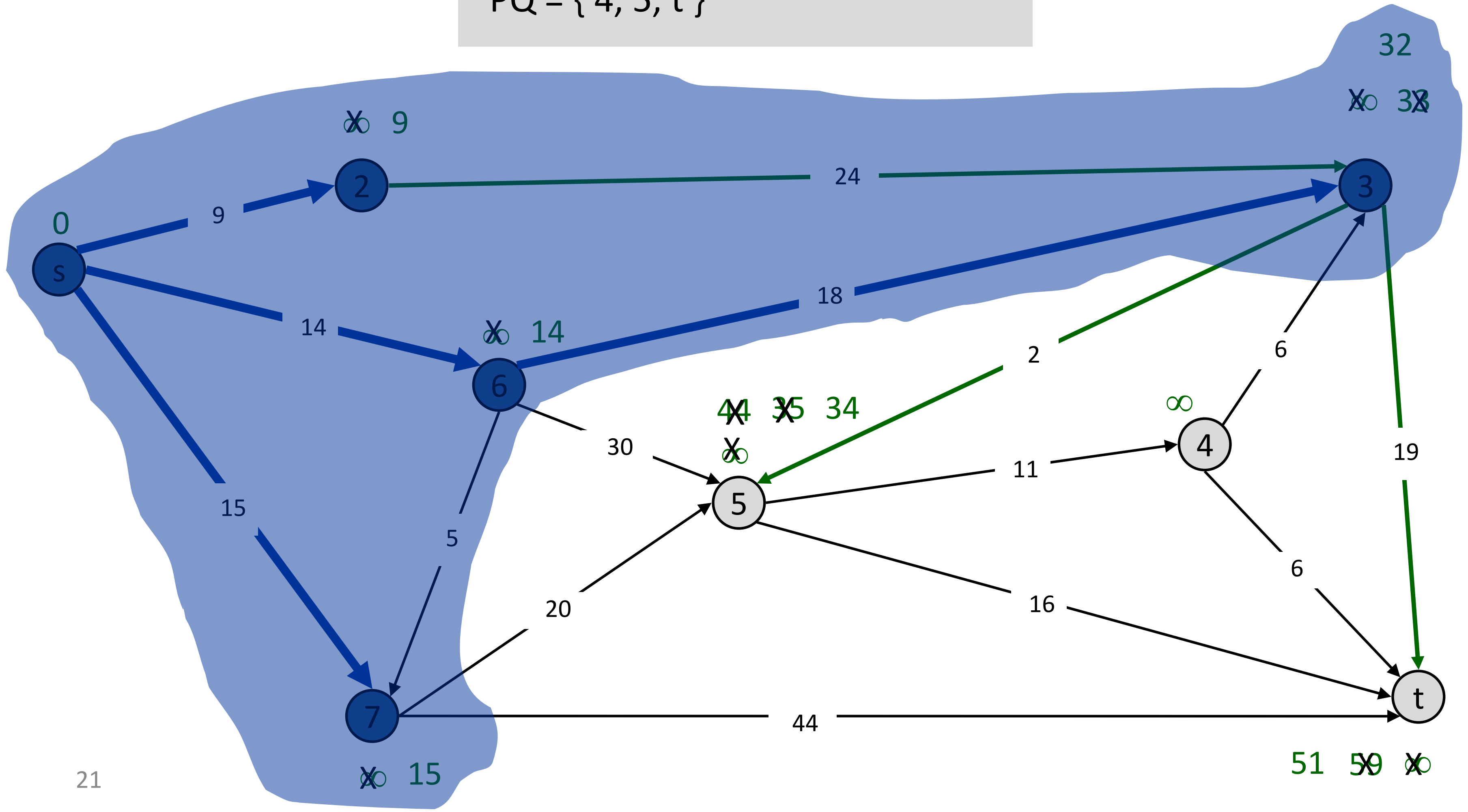

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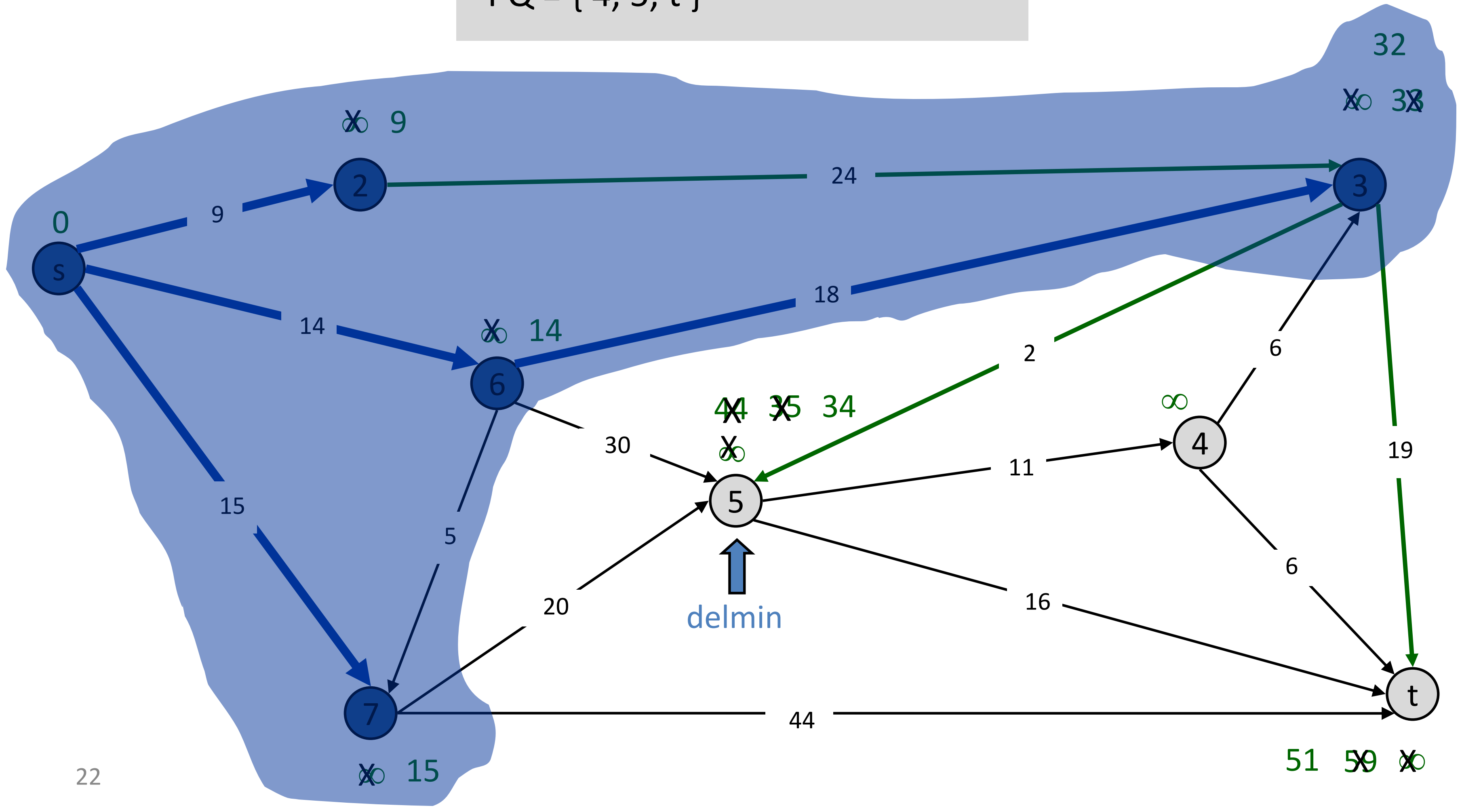

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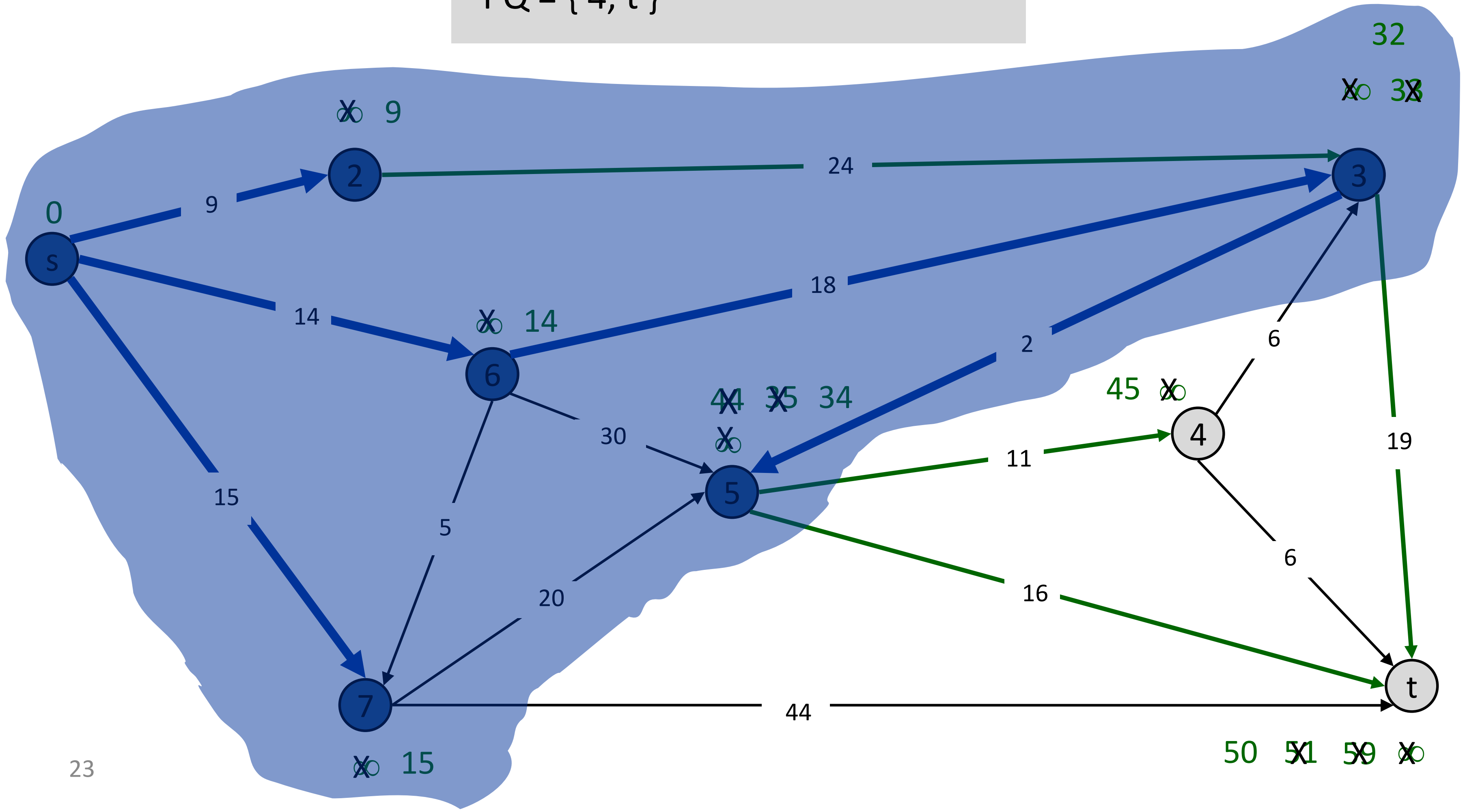

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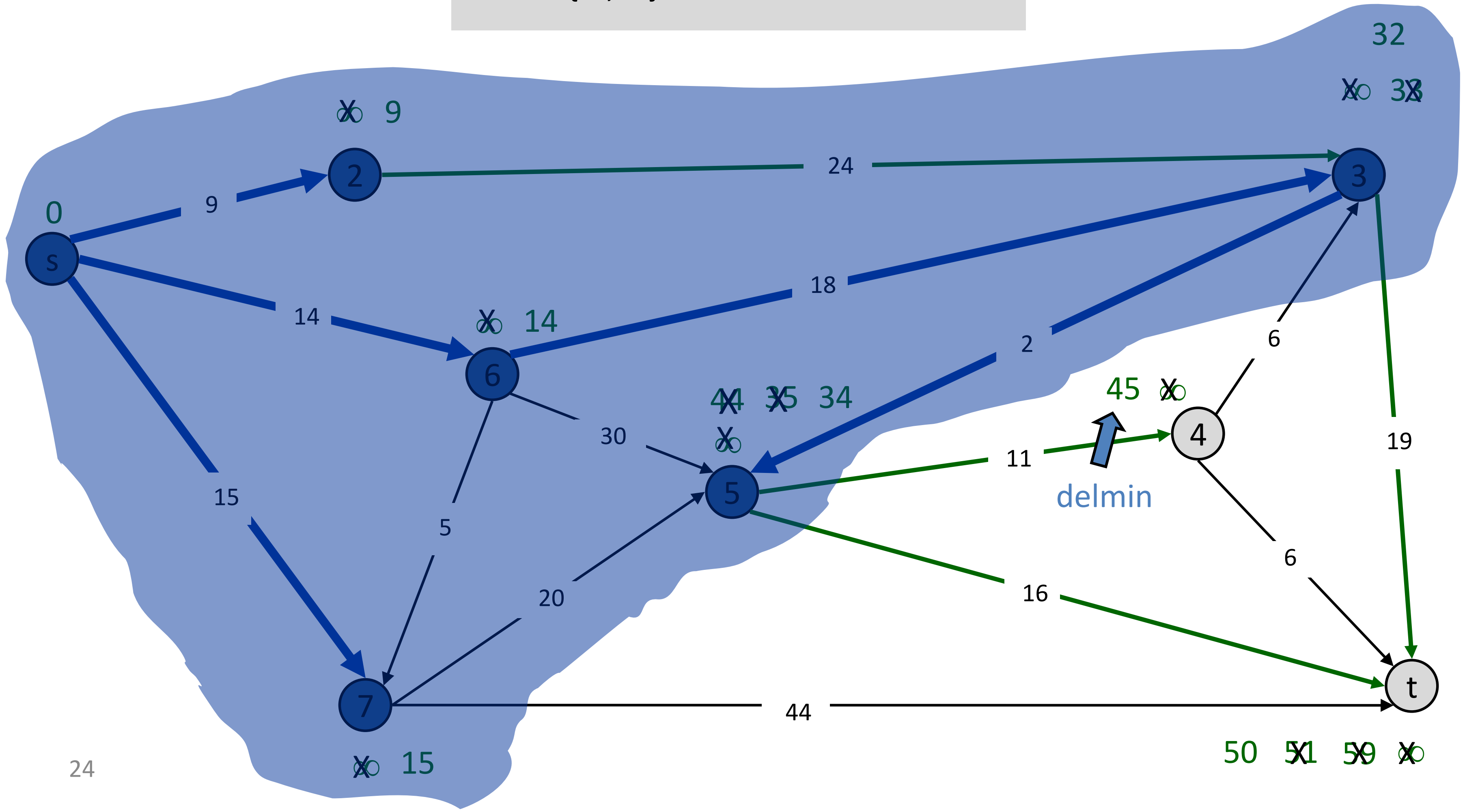

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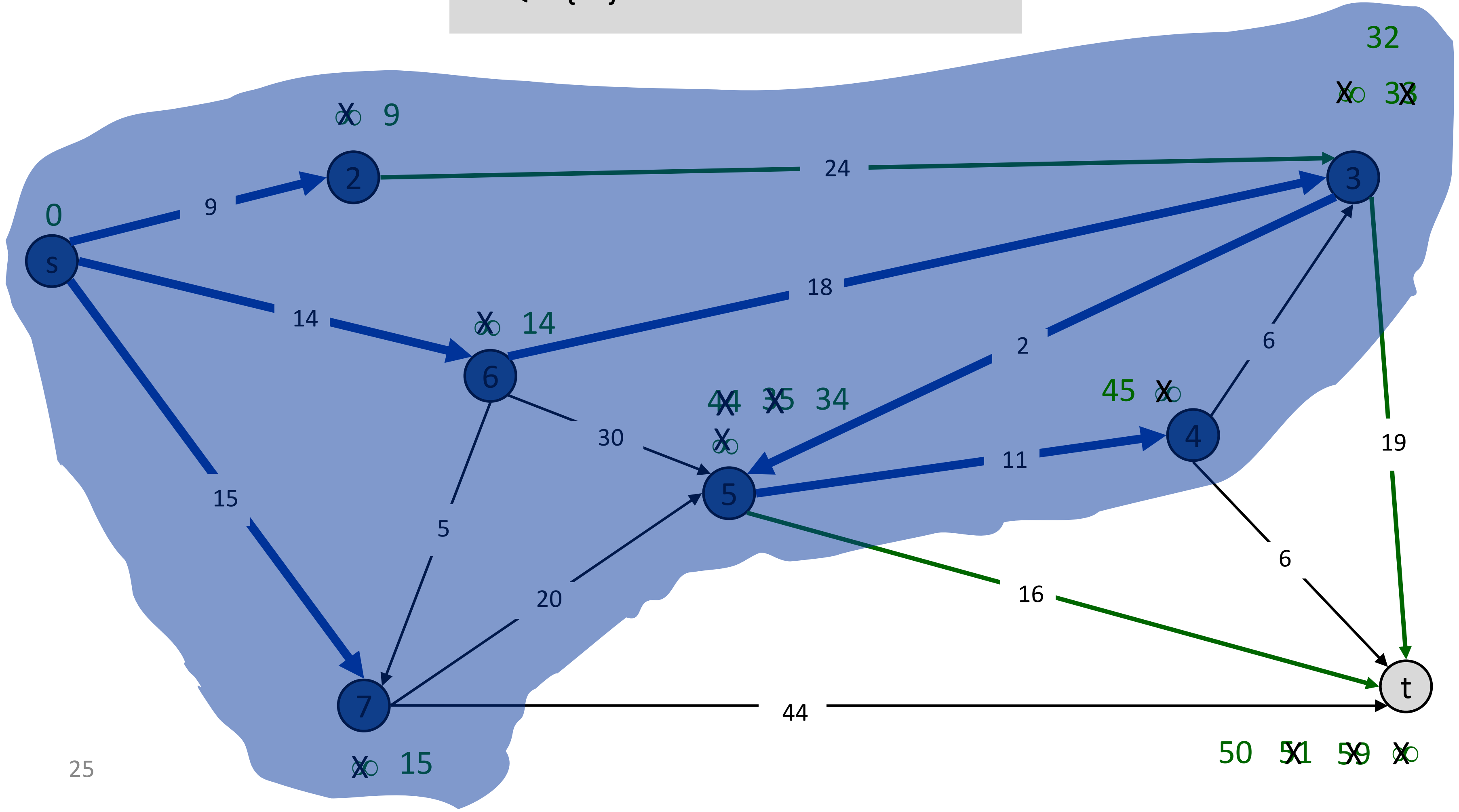

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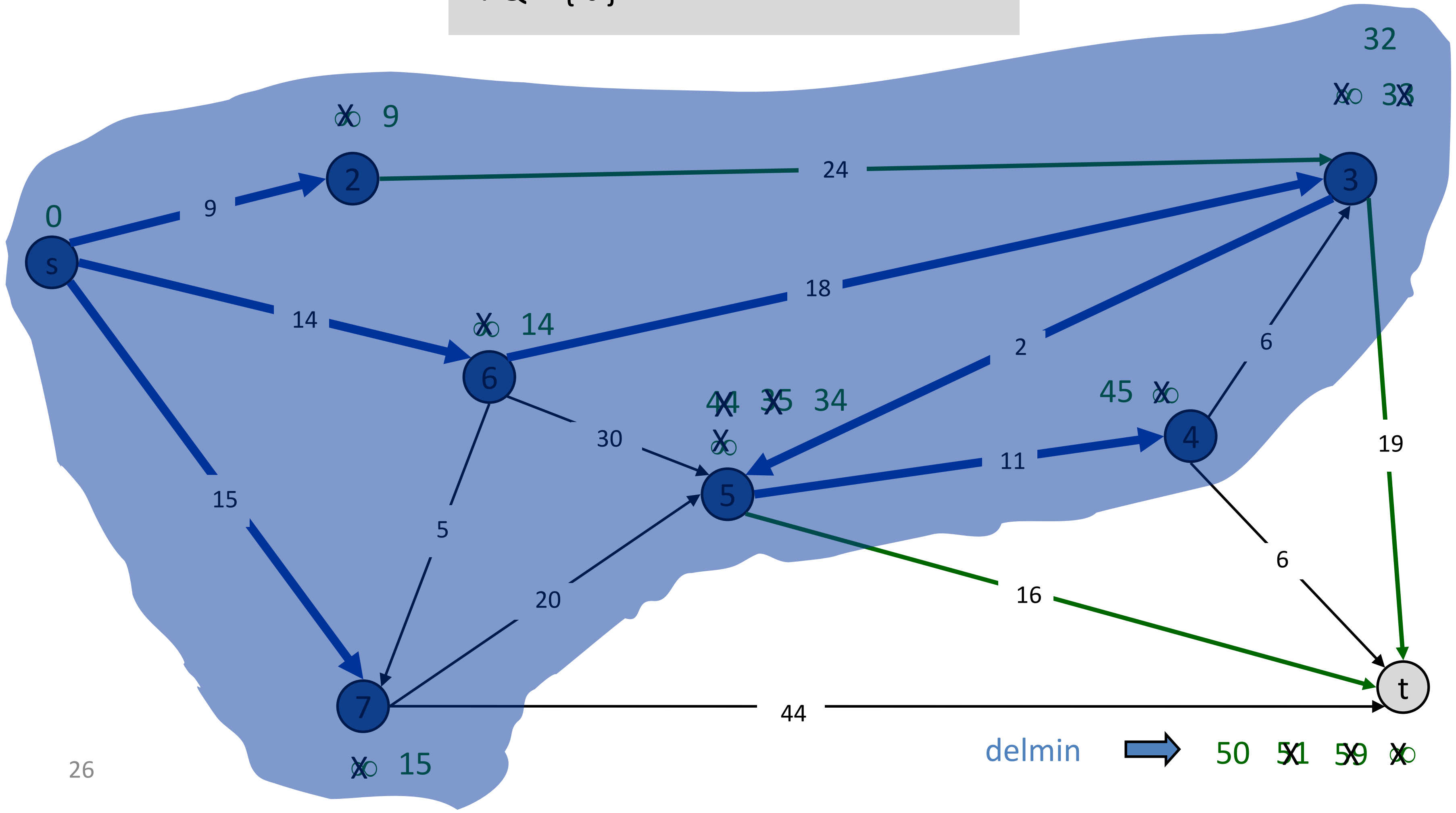

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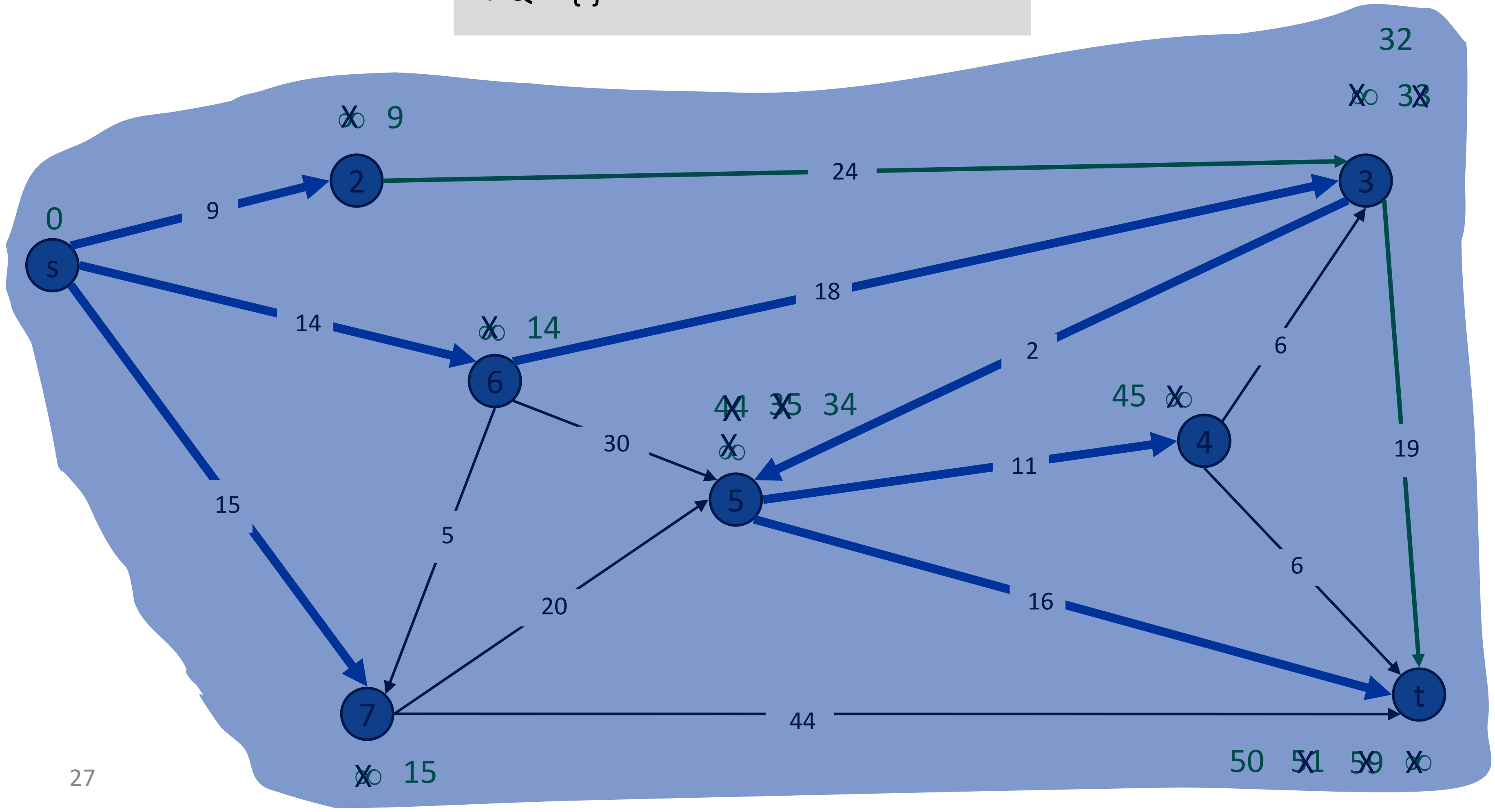

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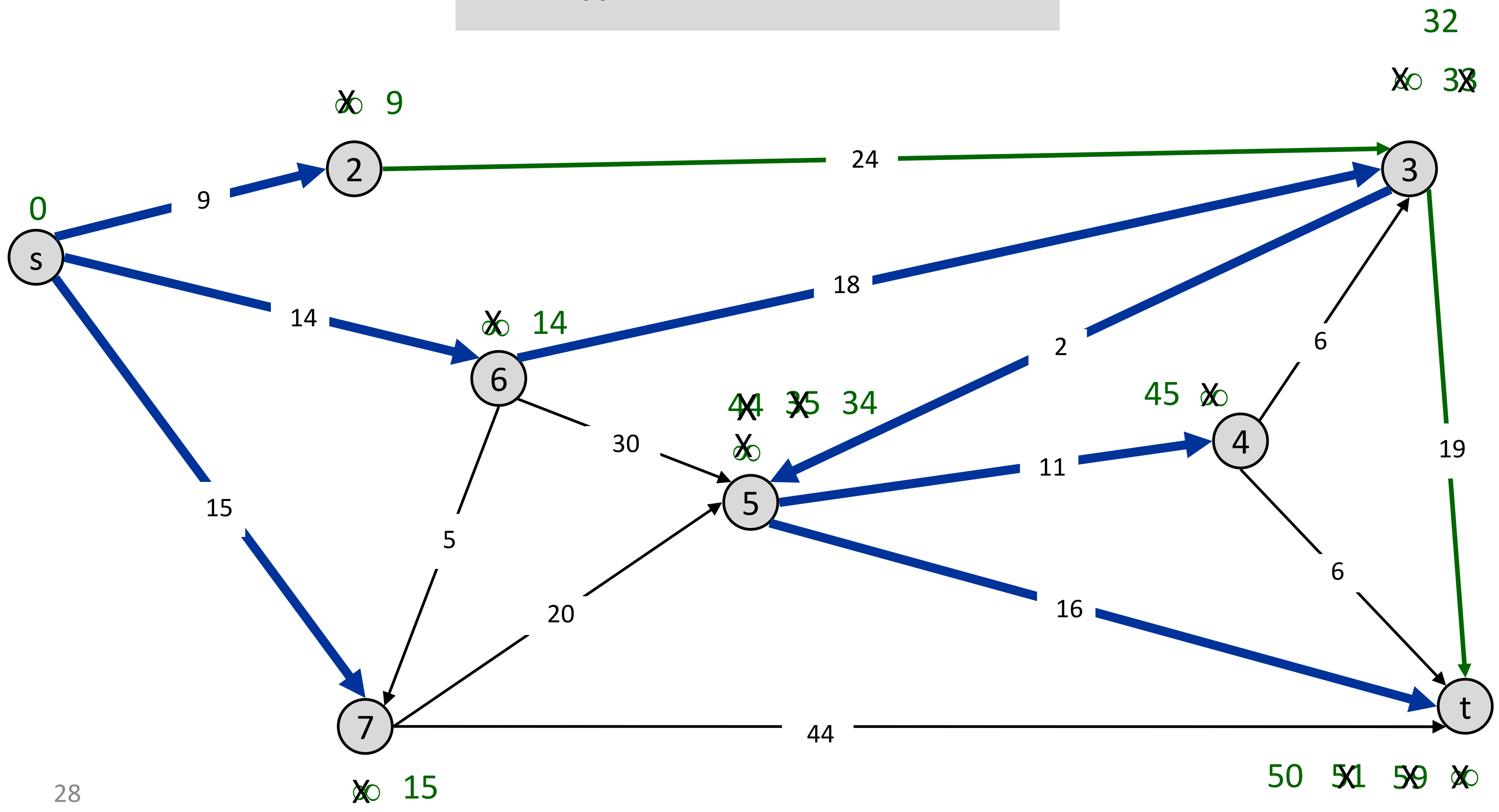
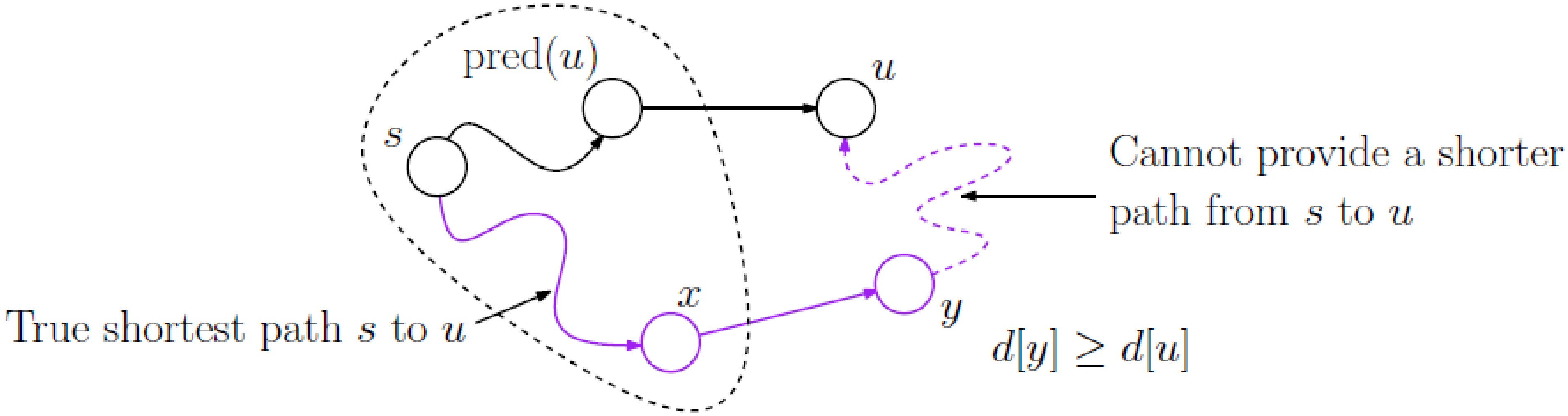


Dijkstra's Algorithm: Proof of Correctness

Let $\delta(s, v)$ = length of true shortest path from s to v.

$d[v]$ is the distance assigned to vertex v by Dijkstra's algorithm.

Lemma: When vertex u is added to S, $\mathrm{d}[u] = \delta(s, u)$.


pred(u)
u
s
Cannot provide a shorter path from s to u
x
y
True shortest path s to u
$d[y] \geq d[u]$

Variants of Dijkstra’s

Vertex weights: There is a cost associated with each vertex. The overall cost is the sum of vertex and/or edge weights on the path.

Single-Sink Shortest Path: Find the shortest path from each vertex to a sink vertex t.

Multi-Source/Multi-Sink: You are given a collection of source vertices {s1, . . . , sk}. For each vertex find the shortest path from its nearest source. (Analogous for multi-sink.)

Multiplicative Cost: Define the cost of a path to be the product of the edge weights (rather than the sum.) If all the edge weights are at least 1, find the single-source shortest path.

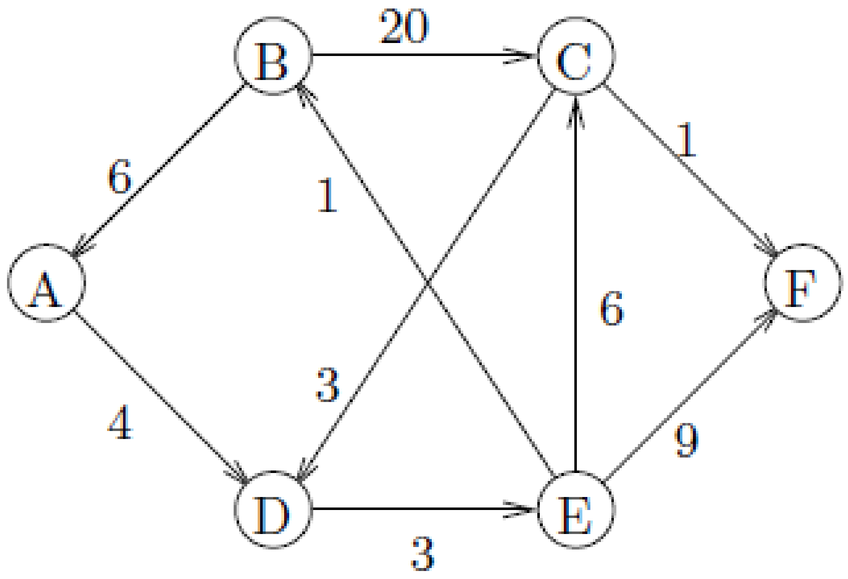
Practice

G = (V, E) is a directed graph with negative weight edges but no negative weight cycles. Which of the following hold for Dijkstra's algorithm on *G*?:

1. The output is correct, and it will run in the same time bound as given in class.
2. The output is correct, but it will take longer than the time bound given in class.
3. It will terminate, but may produce an incorrect result.
4. It may not terminate.

Practice

Give the final d and predecessor values of the vertices obtained by running Dijkstra's algorithm on the directed graph below with source A.



A
B
C
D
E
F
20
6
1
1
6
3
4
9
3